Monotone Pieces of Chains

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Abstract

The problem of decomposing a general polygon into a minimum number of monotone subchains is studied in this paper. A linear-time algorithm for this problem is presented. "Stable" monotone subchains, a concept introduced in this paper, are used to differentiate between stable and unstable chains. A linear-time recognition algorithm for these two classes of chains is presented. The decomposition problem is solved separately for each of these classes.

1 Introduction

A natural problem that arises in the study of geometric computations is that of decomposing objects, of general shape, into pieces, of regular shape, that admit efficient computation. A good example is the art gallery problem proposed by Victor Klee and solved by Chvátal. The result of Chvátal is that any simple polygon with n vertices can be partitioned into at most $\lfloor n/3 \rfloor$ *star-shaped polygons* (and $\lfloor n/3 \rfloor$ is the least upper bound).

Decomposition of two-dimensional objects into regular shapes is a subject that has been vigorously studied with much success. Efficient algorithms for decomposing polygons into star-shaped, convex, triangular pieces exist. Some progress has been made in extending these type of results into the world of three-dimensions. Surprisingly, no one seems to have paid any attention to the one-dimensional case i.e. the decomposition of curves (polygonal

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chains). The results contained in this work may be considered as an investigation of the complexity of decomposing a polygonal chain (which may or may not be simple) into the minimum number of pieces that satisfy certain monotonicity conditions.

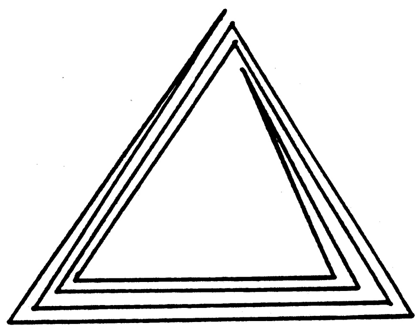
A polygonal chain C can be represented by an ordered sequence of points $\{0, 1, \cdots, n\}$ called the ***vertices*** of C. It can also be represented by an ordered list of ***edges*** $\{e_1, e_2, \cdots, e_n\}$, where e_i is the ***directed*** line segment from vertex i-1 to vertex i. A ***simple polygon*** is a polygon whose edges meet only at their vertices and exactly two edges meet at every vertex.

A chain C given by the sequence of n edges $\{e_1, e_2, \cdots, e_n\}$ is monotone if each edge forms an acute angle ($\leq 90°$) with respect to some fixed direction. Hence we can speak of a ***direction of monotonicity***. We are ready now to state the main problem that we solve with a linear-time algorithm in this paper.

($P^\star$) MINIMUM MONOTONE PARTITION OF SIMPLE CLOSED CHAINS Given the boundary chain C of a simple polygon, find a partition of C into the minimum possible number of monotone subchains. Note that each subchain is permitted to be monotone with respect to a different direction.

It is perhaps difficult to interpret ($P^\star$) on art galleries. However we can descibe an "application" of ($P^\star$) in the cutting of two dimensional patterns out of fabric (cloth, sheet metal, leather, etc.). Consider the cutting table on which the fabric rests that can be moved back and forth in the x-direction. Suppose also that the cutting is always constrained to proceed in the +y direction. Now if we wish to cut out a piece of the fabric in the shape of a simple polygon then the machine can continuously cut a part of the boundary as long as it forms a monotone chain. Cutting any further would require the fabric to be reset on the table. The problem is to ascertain how the cutting can be completed with a minimum number of resets.

Fabric Cutting Theorem:
To cut out a pattern of fabric which is a simple polygon having n vertices, $\lceil n/2 \rceil$ ***resets*** are always sufficient and sometimes necessary.



2 Edge Angles

The idea of using the sequence of edge angles as a partial representation of a polygonal chain was introduced by Shamos in his dissertation. Given a polygonal chain $C = \{e_1, e_2, \cdots, e_n\}$ we associate an ***edge angle*** θ_i with each edge e_i of C.

Lemma 1: A polygonal chain C $= \{e_1, e_2, \cdots, e_n\}$ is monotone if and only if the corresponding sequence of edge angles $\{\theta_1, \theta_2, \cdots, \theta_n\}$ satisfies $\theta_{max} - \theta_{min} \leq 180°$; $\theta_{max} = \max_{1 \leq i \leq n}\{\theta_i\}$; $\theta_{min} = \min_{1 \leq i \leq n}\{\theta_i\}$.

A subchain of a monotone chain is always monotone. A monotone subchain $\{e_i, e_{i+1}, \cdots, e_j\}$ is maximal if both $\{e_i, \cdots, e_{j+1}\}$ and $\{e_{i-1}, \cdots, e_j\}$ are non-monotonic. Note that whenever C is a closed chain, the edge subscripts are to be read modulo (n) and the original e_n is now called e_0.. There can be $O(n)$ maximal monotone chains in a polygon. A maximal monotone subchain can be constructed in linear time by either growing forward first and then reverse ($\{e_i\}_{fr}$) or vice versa ($\{e_i\}_{rf}$). In general, these two chains may be distinct. However, when they are identical, we will have a *stable* chain about e_i. Stable monotone subchains are studied in section 4.

We shall obtain a linear-time solution procedure for (P^*), but first we note that a simple "greedy" procedure which goes around the polygon once to get a set of maximal monotone chains that cover the polygon produces an almost minimum monotone cover of C. Let k_G denote the size of the cover of C found by the greedy algorithm and k_{min} the size of a minimum cover.

Theorem 1: $k_G \leq k_{min} + 1$.

3 Covering a Circle with Arcs

Our problem (P^*) is related to the problem of finding the smallest cardinality arc cover of a circle that has been solved by C.C.Lee and D.T.Lee[1] They describe an algorithm that runs in linear time if the endpoints of the arcs are given in sorted order. It is fairly easy to show that all maximal monotone subchains of C, in sorted order (both left and right endpoints), can be generated in $O(n \log n)$ time using a heap data structure. In section 5, we prove that for a rich class of simple polygons (including convex polygons) that this step can be accomplished in linear time.

4 Stable Monotone Chains

A stable chain was defined in the preliminary sections as a maximal monotone chain grown symmetrically from a single edge e_i so that the forward first chain $\{e_i\}_{fr}$ equals the reverse first chain $\{e_i\}_{rf}$.

Proposition 1: Let C be a simple, closed and polygonal chain given by the edge sequence $\{e_1, e_2, \cdots, e_n\}$. Suppose we start the greedy algorithm at e_δ where $\{e_\delta\}_{fr}$ is a stable chain. Then the greedy algorithm produces an optimal solution (i.e. $k_G = k_{min}$).

[1] C.C. Lee and D.T. Lee, "On a Circle-Cover Minimization Problem", *Information Processing Letters* **18**, (1984), 109-115.

Given an edge angle sequence $\{\theta_i\}$ of a polygonal chain C, we will say that the consecutive subsequence $\{\theta_\alpha, \cdots, \theta_\beta, \cdots, \theta_\gamma\}$ defines a $\Lambda - configuration$ if both $(\theta_\beta - \theta_\alpha)$ and $(\theta_\beta - \theta_\gamma)$ exceed 180°.

Theorem 2: If a polygonal chain C contains a $\Lambda-$ configuration then it contains a stable chain $\{e_\delta\}$ where $\theta_\delta = max\{\theta_\alpha, \cdots, \theta_\gamma\}$.

These two results provide us with a linear-time procedure for solving $(P^\star)$ on polygons containing a $\Lambda-$ configuration.

5 The Case of Polygons with no $\Lambda-$ configuration

Next we will specialize the techniques of section 3, to solve $(P^\star)$ on polygons with no $\Lambda-$ configurations in linear time. Let C denote the boundary chain of a simple polygon with no $\Lambda-$ configuration, described such that the inside of the polygon lies on the left of each directed edge. Let M_k be a maximal monotone subchain $\{e_i, e_{i+1}, \cdots, e_j\}$ of C, with the corresponding sequence of edge angles $\{\theta_i, \theta_{i+1}, \cdots, \theta_j\}$.

Lemma 2: $\theta_{j+1} = \max_{i \leq t \leq (j+1)} \{\theta_t\}$.

Construct the list $\{i, \theta^i_{min}\}$ where $\theta^i_{min} = \min\{\theta_i, \theta_{i+1}, \cdots, \theta_n\}$ $i = 1, 2, ..., n$. This list and above given lemma allows us to run the simple algorithm of section 3 in linear time by using a ***queue*** rather than a ***heap*** data structure and obtain all the maximal monotone chains in $O(n)$ time. These can be given as input to the circle-covering algorithm of C.C.Lee and D.T.Lee to obtain a minimum size cover in linear time.

6 Conclusion

The main result of this paper is an algorithm decomposing a simple and closed polygonal chain into a minimum number of monotone chains. This algorithm is noteworthy for its efficiency since its time-complexity grows only linearly with the number of edges in the chain. The algorithm presented here can be modified slightly to efficiently solve several generalizations of the problem $(P^\star)$.

Decomposition of General Polygonal Chains: $(P^\star)$ was stated as a decomposition problem on the boundary, C, of a simple polygon. We could easily generalize the algorithms to deal with non-simple polygons with no loss in asymptotic efficiency.

Decomposition of Curvilinear Chains: The algorithms of this paper can also be easily be extended to the decomposition of a chain of piecewise continuous and differentiable curves.

Decomposition into ϕ-monotone subchains: A ϕ-monotone subchain is defined to be a subchain whose edges make an angle no more than $\phi/2$ ($\phi \leq 180°$) with a given direction. The results of this paper can be trivially extended to solve for a minimum decomposition of a polygonal (curvilinear) chain into ϕ-monotone subchains in linear time.