ENUMERATING ONE-DIRECTIONAL BLOCKING RELATIONS AND EMBEDDING THEM IN SMALL AREAS ON THE PLANE

Wei-Ping Liu and Ivan Rival

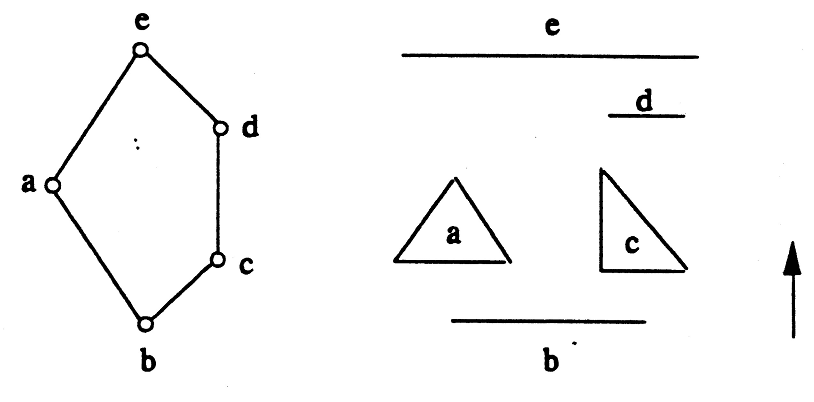
Department of Computer Science
University of Ottawa
Ottawa, Ontatrio, K1N 6N5

Consider a set of disjoint convex figures on the plane and a common direction of motion, say vertical up, for each of the figures. For figures A and B, we say B *obstructs* A, denoted by $A \rightarrow B$, if there is a line joining a point of A to a point of B which follows the moving direction. We write $A < B$ if there is a sequence $A=A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_k=B$. This relation < on the collection P of figures, called a *one-directional blocking relation*, defines an order on P and has been much studied in recent years (cf. [1],[3], [4], [5], [8]). We say the figures with this blocking relation is a *one-directional representation* of the ordered set P (cf. Figure 1). Not every ordered set has such a representation (cf. Figure 2), and, I. Rival and J. Urrutia (1987) have this characterization of one-directional blocking relations.

THEOREM. *An ordered set* P *has a one-directional representation if and only if* P *is a truncated planar lattice.*

A *truncated lattice* is an ordered set obtained from a lattice by removing the top element or bottom element or both.

Such a collection of disjoint convex figures on the plane may model the problem of separating clusters of figures on a computer screen, the problem of guarding collections of of abjets d'art, or even the problem of navigating an iceberg field in our far north.



A diagram of an ordered set P

A one-directional representation of P

Figure 1

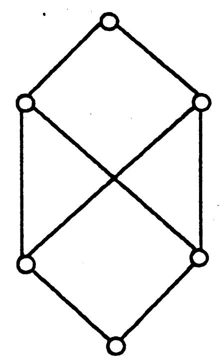
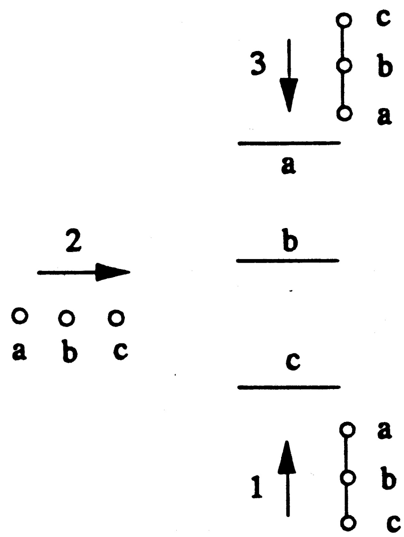


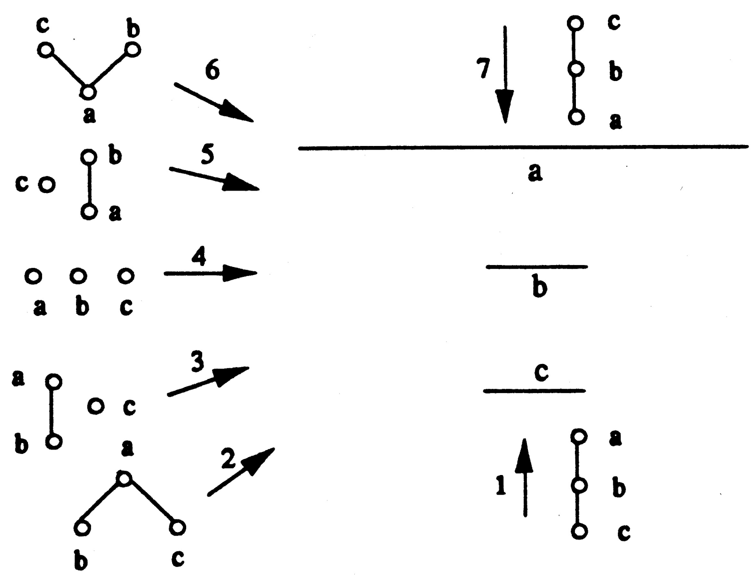
Figure 2

Let R be a one-directional representation of an ordered set P. By changing the direction of movement of the figures (for simplicity, we only consider the case that the angle between the new moving direction and the original is between 0 and π), we may obtain another blocking relation which is a representation of another ordered set Q, with, however, the same underlying set as P. Q is called a *reorientation* of P with respect to the representation R of P, for short, *reorientation.*

Our aim here is two-fold. How many reorientations can P have? What area is needed on the plane to represent P, using say line segments for figures and only integer coordinate locations? A given truncated planar lattice may have different representations, and different representations may, in turn, have different numbers of representations. For example, the representation R_1 of the three-element chain P, a>b>c has three reorientations,



A representation R_1 of P and the three reorientations with respect to it



Another representation R_2 of P and the seven reorientations with respect to it
Figure 3

while the representation R_2 of this three-element chain has 7 reorientations (cf.Figure 3). We adopt the following notation.

Reor(R, P)=|{Q: Q is a reorientation of P with respect to the representation R of P}|.

For brevity, write Reor(P)=max{ Reor(R, P) : R is a representation of P}.

THEOREM 1. *Let* P *be an n-element (truncated) planar lattice. Then,*

$$n\log_4 n-(1+ 1/\ln 4)n \leq \mathrm{Reor}(P) \leq n(n-1) +1.$$

THEOREM 2. (1) *Let* C *be an n-element chain. Then,* Reor(C) =n(n-1)+1.
(2) *Let* A *be an n-element antichain. Then,* Reor(A) ≥ n(n-1).

THEOREM 3. *For arbitrary integers* n *and* m, *there is an n-element planar lattice* P *such that*

$$\mathrm{Reor}(P) \leq 2^{m^2+2} \times n^{1+1/m}.$$

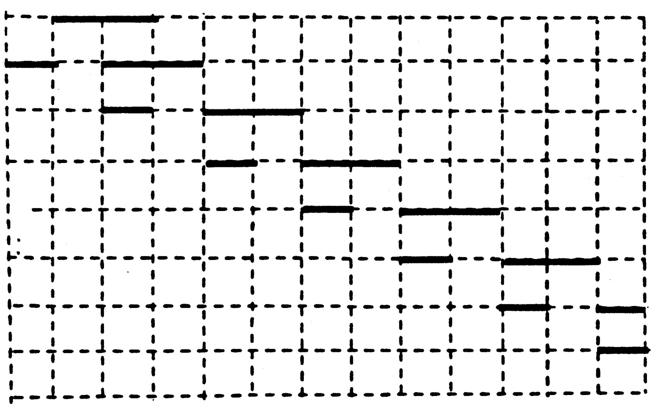
We turn now to representations in which all figures are line segments. Actually, any truncated planar lattice has a representation in which all figures are parallel segments, each with integer length and each of whose ends has integer coordinates. Call such a representation a *parallel segment representation*. What is the smallest area of a parallel segment representation ? (See [2], [6], [7] for a discussion of small area representations of diagrams of orders and planar graphs using integer coordinates.)

For a parallel segment representation R of a truncated planar lattice P, let Area(R,P) stand for the area of the smallest upright rectangle enclosing R . For example, for the representation R of the ordered set P illustrated in Figure 5, Area(R, P)=13 × 7=91, and for the representation R of Q illustrated in Figure 6, Area(R, Q)=55.

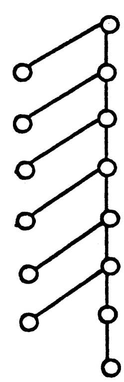
THEOREM 4. *Let* P *be an n-element truncated planar lattice. Then*

min{Area(R,P): R is a parallel segment repesentation of P} ≤ n(n-1)/2.

and this is best possible (Figure 5).



A parallel segment representation of P



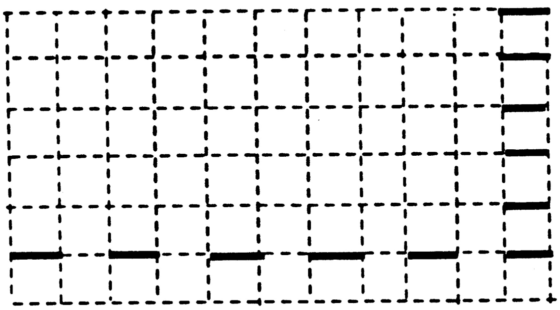
A diagram P

Figure 5

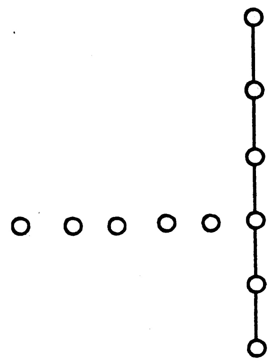
Although Reor(R,P) is big for the representations in Theorem 2, the representations have 'large' areas. If the segments of a parallel segmant representation R of an ordered set P are confined in the smallest area, is Reor(R, P) necessarily small ? It turns that it is not the case.

THEOREM 5. *For any* n, *there is an n-element truncated planar lattice* P *such that some parallel segment representation* R *of* P *satisfies the following conditions*:

1) Area(R, P)=min{Area(R_1, P) : R_1 is a parallel segment representation of P };
2) Reor(R, P)=O(n^2 / lnn) (Figure 6).



A parallel segment representation of Q



A diagram Q

Figure 6

We may enumerate the blocking relations in space, too. That is, given a representation in space of an ordered set, how many reorientations are there by changing the direction of movement of the figures, each, again, assigned the same direction ? Even for the simple ordered set P consisting of n disjoint noncomparable two-element chains has at least 2^n reorientations with respect to some representation of P.

REFERENCES

[1] R. Dawson (1984) On removing a ball without disturbing the others. *Mat. Mag.* 57, 27-30

[2] H. de Fraysseix, J. Pach and R. Pollack (1988) Smallest sets supporting Fary embeddings of planar graphs. *Proc. 20th ACM Symp. Th. of Comp.* 426-433.

[3] M. Mansouri and G. T. Toussaint (1985) Translation queries for convex polygons. *Proc.* IASTED *Internat. Sympos. Robotics and Automation.*, Lugano, Switzerland.

[4] R. J. Nowakowski, I. Rival and J. Urrutia Representing orders on the plane by translating points and lines. *Discrete Math.* to appear.

[5] I. Rival and J. Urrutia (1987) Representing orders on the plane by translating convex figures. *Order* 4, 319-339.

[6] G. Di Battista, R.Tamassia and I. G. Tollis (1989) Area Requirement and Symmentry Display in Drawing Graphs. *Proc. 5th* ACM *Symposium on Computational Geometry*.

[7] R. Tamassia and I. G. Tollis (1986) A United Approroach to Visibility Representations of Planar Graphs. *Discrete Computational Geometry,* 1, 321-341.

[8] S. Wismath (1985) Charaterizing bar line-of-sight graphs, *Proc. 1st ACM Symp. Computational Geometry,* pp. 147-152.