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Spherical Orders, Planar Lattices and Obstruction Graphs in Abstract Convection Systems

A *convection system* is defined as a partially ordered group T, called *time model*, acting on a set D, called *domain*, and where all stabilizers are trivial. For $a \in D$, the orbit $\{ta : t \in T\}$ then represents the time-dependent positions of a particle moving in the convection system. Orbits possess a natural order structure isomorphic to that of T, where $a \leq b$ if and only if the (unique) $t \in T$ such that $ta = b$ is positive (i.e., $e \leq t$ for the unit element e of T).

Steady fluid flows studied in physics are the natural examples of convection systems. Here D is a subset of some $\mathbb{R}^n$, $T = \mathbb{R}$ and the orbits, called streamlines, are described by differential equations. The purpose of this presentation is to show that certain combinatorial properties of convection systems are independent of the topology and geometrical nature of D, and are order-theoretical in essence.

As each orbit is ordered, so is the entire domain D, which is partitioned into orbits. Points belonging to different orbits are incomparable. For $a \in D$, the *ray* from a is $[a, \rightarrow) = \{b \in D : a \leq b\} = \{ta : e \leq t\}$. For $a, b \in D$ the interval $[a, b] = \{x \in D : a \leq x \leq b\}$ may be empty; if it is not empty, it can be called the *segment* from a to b, and denoted $\overline{ab}$.

Perhaps the two simplest natural examples of convection systems are steady fluid flows with $D \subseteq \mathbb{R}^2$. In the *uniform flow*, D is the entire plane $\mathbb{R}^2$, and the motion (group action) is described by $t \times (x, y) = (x, t+y)$ for all $t \in \mathbb{R}$, $(x, y) \in \mathbb{R}$. The orbits here are straight vertical lines. In the *central flow*, on the other hand, $D = \mathbb{R}^2 \setminus \{(0,0)\}$, and $t \times (x, y) = (k^t x, k^t y)$, where k is any fixed real number greater than 1. Now the orbits are the straight lines drawn from the origine (minus the origine itself). Certain combinatorial differences between uniform and central flows, first described by Foldes, Rival and Urrutia [1], are indeed at the origin of the questions considered here.

We are concerned with obstructions that may occur between objects, in the sense that the motion of an object released in a specific position in the convection system, and "carried by the abstract fluid", may be obstructed by another object occupying a specific position. Formally, some set O of subsets of D need to be specified as *objects*. Depending on how inclusive the object class O is, the possible obstruction relationships that may arise will vary.

These obstruction relationships among the objects can be studied in terms of a directed graph. By a *collection* of objects we mean a finite, pairwise disjoint set S of objects. The corresponding *obstruction graph* is the directed graph with vertex set S and where there is a directed edge from A to B whenever $A \neq B$ and for some positive t, $tA \cap B \neq \emptyset$ (then B is said to *obstruct* A). The absence of directed cycles in the obstruction graph is a property of particular interest for the purposes of planning the motions of disassembly of clustered objects, it indeed corresponds to a condition of sequential separability. In this acyclic case, the transitive closure of the obstruction graph is a directed comparability graph, defining a partial order on the collection of objects, called the *obstruction order* (or *blocking relation*). $A \leq B$ in this order if and only if there is a directed path from A to B in the obstruction graph. The question of what obstruction graphs and orders can or cannot arise in a given convection system has been investigated in a number of cases [1, 5, 2, 3], in particular in the case of uniform and central flows in two dimensions. To state the results pertaining to these latter, we need to recall two definitions:
A partially ordered set with top and bottom elements Max, Min is a *planar lattice*, resp. a *spherical order*, if its coverring diagram can be drawn without intersecting edges on the plane, resp. on the sphere, so that along each edge the y-coordinate, resp. the spherical latitude, continuously increases. After removal of any subset of {Max, Min} such ordered sets are called *truncated*.

Proposition 1 *(Guibas, Yao [4] and Rival, Urrutia [5]) The obstruction graph of a collection of convex polygons in a two-dimensional uniform flow is acyclic. The obstruction order is a truncated planar lattice, and all truncated planar lattices arise this way.*

Of particular interest are objects that are *convex* in the order of D, i.e., objects A that contain the intervals $[a, b]$ (equivalently, the segments $\overline{ab}$), for $a, b \in A$. Given a non-convex set A, there is a $b \in D \setminus A$, such that A and $\{b\}$ mutually obstruct each other, and the existence of such b is characteristic of non-convex sets. Indeed, studies of obstruction and separability have been mostly concerned with objects that are convex in some sense. In the sequel we shall also assume the convexity of objects (always meant hereafter in the order-theoretical sense).

Note that if $A \subseteq D$ is convex, then so is tA for any $t \in T$, i.e., convex sets do not lose their convexity by convection. When considering any set of objects O, we shall always assume additionally:

(01) If $A, B \in O$, then the convex hull of $A \cup B$ is again an object.

(02) For every $A \in O$, there is a positive $t \in T$ such that $A \cap tA = \emptyset$.

Postulate **(02)** is a generalization of compactness. As for the convex hull of any set C, recall that it is the union of C and of all the segments having both endpoints in C.

Proposition 2 *(Foldes, Rival and Urrutia [1]) In a two-dimensional central flow, all obstruction orders arising from collections of polygonal objects are truncated spherical, and all truncated spherical orders arise this way.*

The following Propositions are independent of any planar or spherical geometry:

Proposition 3 *Given any convection system and object class, the following two conditions are equivalent:*
(i) directed cycles of arbitrarily large size (chordless) arise as obstruction graphs,
(ii) all spherical orders arise as obstruction orders.

Proposition 4 *Given any convection system and object class, the following two conditions are equivalent:*
(i) directed paths of arbitrarily large length (chordless) arise as obstruction graphs,
(ii) all planar lattices arise as obstruction orders.

The equivalent conditions of Proposition 4 hold in the two-dimensional uniform flow with convex polygonal objects, according to Rival and Urrutia [5]. The stronger equivalent conditions of Proposition 3 do not hold here but they do hold in the two-dimensional central flow with order-theoretically convex polygonal objects (Foldes, Rival and Urrutia [1]). In both these cases the time model is the real line $\mathbb{R}$. For a different example, let $T = \mathbb{R}^2$, with $(x, y) \leq (x', y')$ meaning $x \leq x'$ and $y \leq y'$, act on $D = \mathbb{R}^2$, by $(t_1, t_2)(x, y) = (t_1 + x,\ t_2 + y)$, and let the objects be order-theoretically convex polygons. This corresponds to a case of non-deterministic multi-directional convection, such as discussed in [2]. There again, the conditions of Proposition 4 hold, but those of Proposition 3 do not.

References

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