Barycentric triangulation of generalized maps

Pascal Lienhardt

C.N.R.S.
Département d'Informatique
Université Louis Pasteur
7, rue René Descartes
67084 Strasbourg Cedex, France
Tel : 88 41 63 46

Extended abstract

N-dimensional generalized maps, or *n-G-maps*, are presented in [1] and [2], for the representation of the topology of subdivisions of n-dimensional topological spaces, orientable or not orientable, with or without boundaries (informally, a subdivision of a topological space is a partition of this space into vertices, edges, faces, volumes, ..., i.e. into cells with dimension 0, 1, 2, 3 ...; see also the notion of *cell-tuple structure* presented in [3], which is very close to the notion of n-G-map) . The notion of n-G-map is an extension of the notion of combinatorial map, defined in [4], and studied by many authors ([5], [6], [7], [8], ...) .
The distinction between *topology* and *embedding* of a geometric object allows to differentiate among problems which raise up in Geometric Modeling, and more generally in Computational Geometry, and sometimes allows to define a hierarchization of these problems (for instance in animation of articulated objects, for computing Voronoï diagrams : see for instance [9]) .

N-dimensional simplices and *barycentric triangulation* of a subdivision are classical notions in algebraic topology . A 1-simplex is an edge, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, ... The barycentric triangulation T of a subdivision M of an n-dimensional topological space is (informally) a particular decomposition of this subdivision into n-simplices, such that the topological spaces subjacent to T and M are homeomorphic (for a more complete definition, see [10] and [11]) . We here study for n-G-maps the corresponding combinatorial notions, and we define :

- *simplex n-G-maps*, which make a particular class of n-G-maps ; important properties of simplex n-G-maps are presented, and we describe a method for constructing simplex n-G-maps ;
- the *barycentric triangulation* G^T of an n-G-map G, which is an n-G-map deduced from G, and made up by the gathering of simplex n-G-maps . An important result is that number of boundaries and orientability are invariant by barycentric triangulation .

I think that simplex n-G-maps and barycentric triangulation may simplify the study of subdivisions of topological spaces, whose topology may be defined by an n-G-map .

The notion of n-G-map is here briefly reminded (for a more complete study, see [1] and [2]) .
An *n-G-map* G is defined by an (n+2)-tuple $G = (B, \alpha_0, \alpha_1, \ldots \alpha_n)$, such that (Fig. 1) :

- B is a finite, non empty set of *darts* ;
- for each i, $0 \leq i \leq n-1$, α_i is an involution without fixed points, α_n is an involution, such that : for each i, j, $0 \leq i < i+2 \leq j \leq n$, $\alpha_i\alpha_j$ is an involution (i.e. $\alpha_i\alpha_j = \alpha_j\alpha_i$) .

An n-G-map $G = (B, \alpha_0, \alpha_1, \ldots \alpha_n)$, such that α_n is without fixed points, is said *without boundaries* . For each set Φ of permutations of B, let $<\Phi>$ be the *group* of permutations of B generated by Φ . For each dart b of B, $<\Phi>(b) = \{\phi(b), \phi \in <\Phi>\}$ is the *orbit* of b relatively to group Φ . $<\alpha_0, \alpha_1, \ldots \alpha_n>(b)$ is the *connected component* of G incidental to b .
Let $G = (B, \alpha_0, \alpha_1, \ldots \alpha_n)$ be an n-G-map without boundaries . The n-G-map G*, *dual* of G, is defined by : $G^* = (B, \alpha_n, \alpha_{n-1}, \ldots \alpha_0)$.
An *n-map* C is defined by an (n+1)-tuple $C = (B, \alpha_0, \alpha_1, \ldots \alpha_{n-1})$, such that :

- B is a finite, non empty set of darts ;
- for each i, $0 \leq i \leq n-2$, α_i is an involution, α_{n-1} is a permutation, such that : for each i, j, $0 \leq i < i+2 \leq j \leq n-1$, $\alpha_i\alpha_j$ is an involution .

An n-G-map $G = (B, \alpha_0, \alpha_1, \ldots \alpha_n)$ without boundaries is said to be *orientable* if and only if the n-map $(B, \alpha_n\alpha_0, \alpha_n\alpha_1, \ldots \alpha_n\alpha_{n-1})$ has two connected components (cf. [1] for the definition of orientability in the general case) .
In order to simplify, we assume the n-G-maps are connected .

1.– Simplex n-G-maps .

Definition 1 : Let $G = (B, \alpha_0, \alpha_1, ...\alpha_{n-1})$ be an (n-1)-G-map ($n \geq 1$) . Let $b \in B$, and let i, j be such that $0 \leq j \leq i \leq n-1$. We define :

$b_0 = b$;
$b_1 = b\alpha_i$;
$b_2 = b\alpha_{i-1}\alpha_i$;
...
$b_k = b\alpha_{i-k+1}\alpha_{i-k+2}...\alpha_{i-1}\alpha_i$;
...
$b_{i-j+1} = b\alpha_j\alpha_{j+1}...\alpha_{i-1}\alpha_i$.

G has *property of (j,i)-disjunction* if and only if : $\forall\, b \in B$:

$\forall\, k,l\ (0 \leq k,l \leq i-j+1,\ k \neq l) : <\alpha_j,\alpha_{j+1},...\alpha_{i-1}>(b_k) \cap <\alpha_j,\alpha_{j+1},...\alpha_{i-1}>(b_l) = \varnothing$.

G has *property of disjunction* if and only if G has property of (j,i)-disjunction, for all i, j, $0 \leq j \leq i \leq n-1$.

G has *property of (j,i)-union* if and only if : $\forall\, b \in B$:

$(\cup <\alpha_j,\alpha_{j+1},...\alpha_{i-1}>(b_k))_{k=0,...i-j+1} = <\alpha_j,\alpha_{j+1},...\alpha_i>(b)$.

G has *property of union* if and only if G has property of (j,i)-union, for all i, j, $0 \leq j \leq i \leq n-1$.

G has *property of (j,i)-partition* (resp. *partition*) if and only if G has properties of (j,i)-disjunction (resp. disjunction) and of (j,i)-union (resp. union) (Fig. 2) .

Definition 2 : Let $G = (B, \alpha_0, \alpha_1, ...\alpha_{n-1})$ be an (n-1)-G-map ($n \geq 2$) . G is said to be *quasi-triangular* if and only if, for all i ($0 \leq i \leq n-2$) : $(\alpha_i\alpha_{i+1})^3$ = identity on B .

Theorem 1 : An (n-1)-G-map G ($n \geq 1$) has property of partition if and only if G has property of (0,n-1)-disjunction and G is quasi-triangular ($n \geq 2$) .

Theorem 2 : An (n-1)-G-map G ($n \geq 1$) has property of partition if and only if the (n-1)-G-map G* dual of G has property of partition .

Theorem 3 : Let G and G' be two (n-1)-G-maps, both having property of partition . Let $b \in B$, and $b' \in B'$. Then an isomorphism Φ exists between G and G', such that $\Phi(b) = b'$ (i.e. two rooted n-G-maps having property of partition are isomorphic) .

Definition 3 : Let $G = (B, \alpha_0, \alpha_1, ...\alpha_n)$ be an n-G-map ($n \geq 2$) . G is a *simplex n-G-map* if and only if (Fig. 2) :

– α_n is the identity on B ;
– $\partial(G) = (B, \alpha_0, \alpha_1, ...\alpha_{n-1})$ has property of partition .

Theorem 4 : Two rooted simplex n-G-maps are isomorphic .

Theorem 5 : A simplex n-G-map has exactly one boundary, and is orientable .

A method for constructing simplex n-G-maps, given simplex (n-1)-G-maps, will be described in the final paper (for a more complete study, and proofs of theorems, see [12]) .

2.– Barycentric triangulation .

Definition 7 : Let $G = (B, \alpha_0, \alpha_1, ...\alpha_n)$ be a connected n-G-map . Let i be such that $1 \leq i \leq n$. The n-G-map $G^T = (B^T, \alpha^T_0, \alpha^T_1, ...\alpha^T_n)$, called *barycentric i-triangulation of G*, is defined by (Fig. 3) : to each dart b of B is associated a simplex i-G-map $S^b = (B^b, \alpha^b_0, \alpha^b_1, ...\alpha^b_i)$, where a dart b^b is distinguished . Then :

– $B^T = (\cup B^b)_{b \in B}$;
– $\forall\, j,\ 0 \leq j \leq i-1$: the restriction of α^T_j to B^b is equal to α^b_j ;

– $\forall$ j, $i+1 \leq j \leq n$, and $\forall b \in B$: let S^b and $S^{b\alpha_j}$ be the simplex i-G-maps associated respectively to b and to $b\alpha_j$; an isomorphism ϕ exists between S^b and $S^{b\alpha_j}$, such that $\phi(b^b) = (b\alpha_j)^b$ (ϕ is unique) ; $\forall b' \in B^b : b'\alpha^T_j = \phi(b')$; $\forall b' \in B^{b\alpha_j} : b'\alpha^T_j = \phi^{-1}(b')$.
– α^T_i : for each dart b of B, we define :

$b_0 = b^b$;
$b_1 = b^b\alpha^b_{i-1}$ ($= b^b\alpha^T_{i-1}$) ;
$b_2 = b^b\alpha^b_{i-2}\alpha^b_{i-1}$ ($= b^b\alpha^T_{i-2}\alpha^T_{i-1}$) ;
...
$b_i = b^b\alpha^b_0 ... \alpha^b_{i-2}\alpha^b_{i-1}$ ($= b^b\alpha^T_0 ... \alpha^T_{i-2}\alpha^T_{i-1}$) .

and :

$b'_0 = (b\alpha_i)^b$;
$b'_1 = (b\alpha_{i-1})^b\alpha^T_{i-1}$;
$b'_2 = (b\alpha_{i-2})^b\alpha^T_{i-2}\alpha^T_{i-1}$;
...
$b'_i = (b\alpha_0)^b\alpha^T_0 ... \alpha^T_{i-2}\alpha^T_{i-1}$.

For each j ($0 \leq j \leq i$), an isomorphism ϕ_j exists between $<\alpha^T_0, \alpha^T_1, ... \alpha^T_{i-2}>(b_j)$ and $<\alpha^T_0, \alpha^T_1, ... \alpha^T_{i-2}>(b'_j)$, such that $\phi_j(b_j) = b'_j$ (ϕ_j is unique) . Let Φ be the "product" of these isomorphisms . Then, for each dart b' of B^b : $b'\alpha^T_i = \Phi(b')$.

Theorem 7 : The barycentric i-triangulation of an n-G-map is an n-G-map .

Theorem 8 : Let G and G' be two n-G-maps, such that G and G' are isomorphic . Let G^T and G'^T be two n-G-maps respectively deduced from G and G' by barycentric i-triangulation . Then G^T and G'^T are isomorphic .
Corollary : Two barycentric i-triangulations of an n-G-map are isomorphic.

Theorem 9 : Number of boundaries and orientability are invariant by barycentric i-triangulation .

Definition 8 : The barycentric n-triangulation of an n-G-map G is called *barycentric triangulation* of G .

Références .

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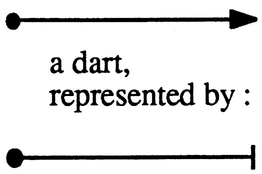
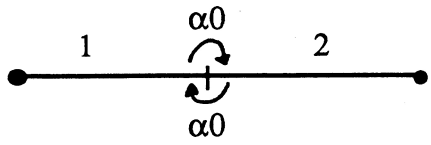
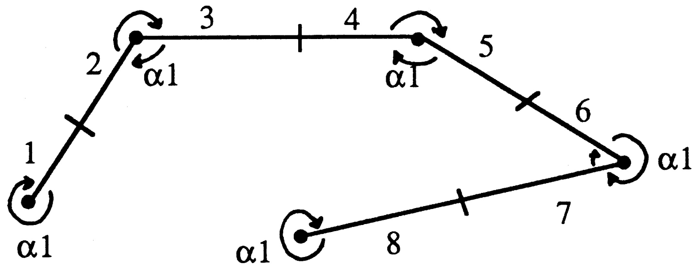




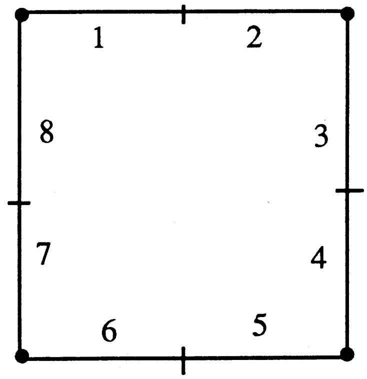
$\alpha 0 = \{1, 2\}$

n = 0 : a non oriented edge



$\alpha 0 = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$
$\alpha 1 = \{\{1\},\{2,3\},\{4,5\},\{6,7\},\{8\}\}$

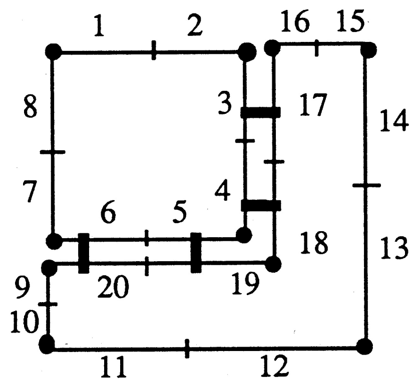
n = 1 : a simple elementary path



$\alpha 0 = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\}\}$
$\alpha 1 = \{\{1,8\},\{2,3\},\{4,5\},\{6,7\}\}$

n = 1 : a simple elementary cycle

n = 2



$\alpha 2 = \{\{1\}, \{2\}, \{3, 17\}, \{4, 18\}, \{5, 19\}, \{6, 20\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13\}, \{14\}, \{15\}, \{16\}\}$

the tie $\alpha 2$ is symbolized by a thick line
if $b\alpha 2 = b$, then $\alpha 2$ is not represented

n =3

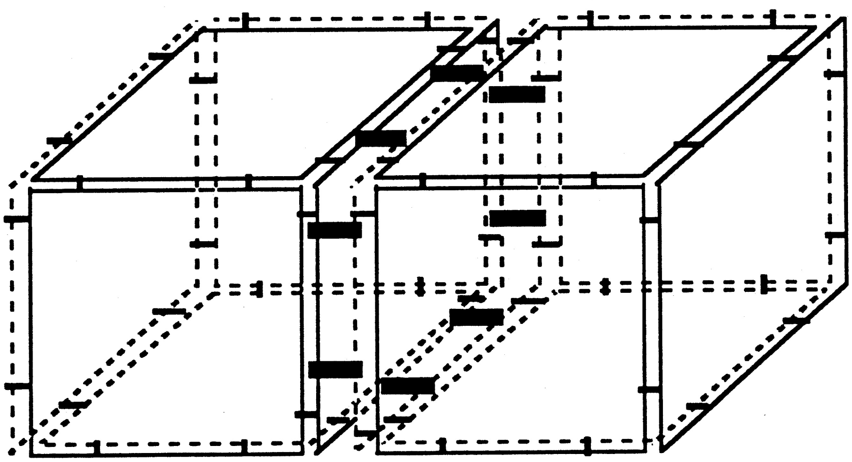


Fig. 1 : 0- , 1- , 2- and 3-G-maps .

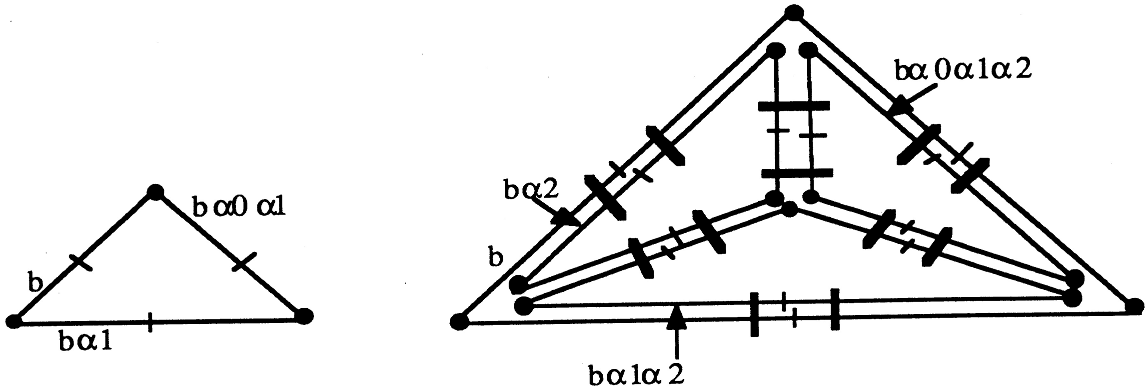


Fig. 2 : left, a 1-G-map having property of partition. Each dart b, $b\alpha_1$, $b\alpha_0\alpha_1$ is incidental to a distinct connected component of (B, α_0). Right, a 2-G-map having property of partition. Each dart b, $b\alpha_2$, $b\alpha_1\alpha_2$, $b\alpha_0\alpha_1\alpha_2$ is incidental to a distinct connected component of (B, α_0, α_1).

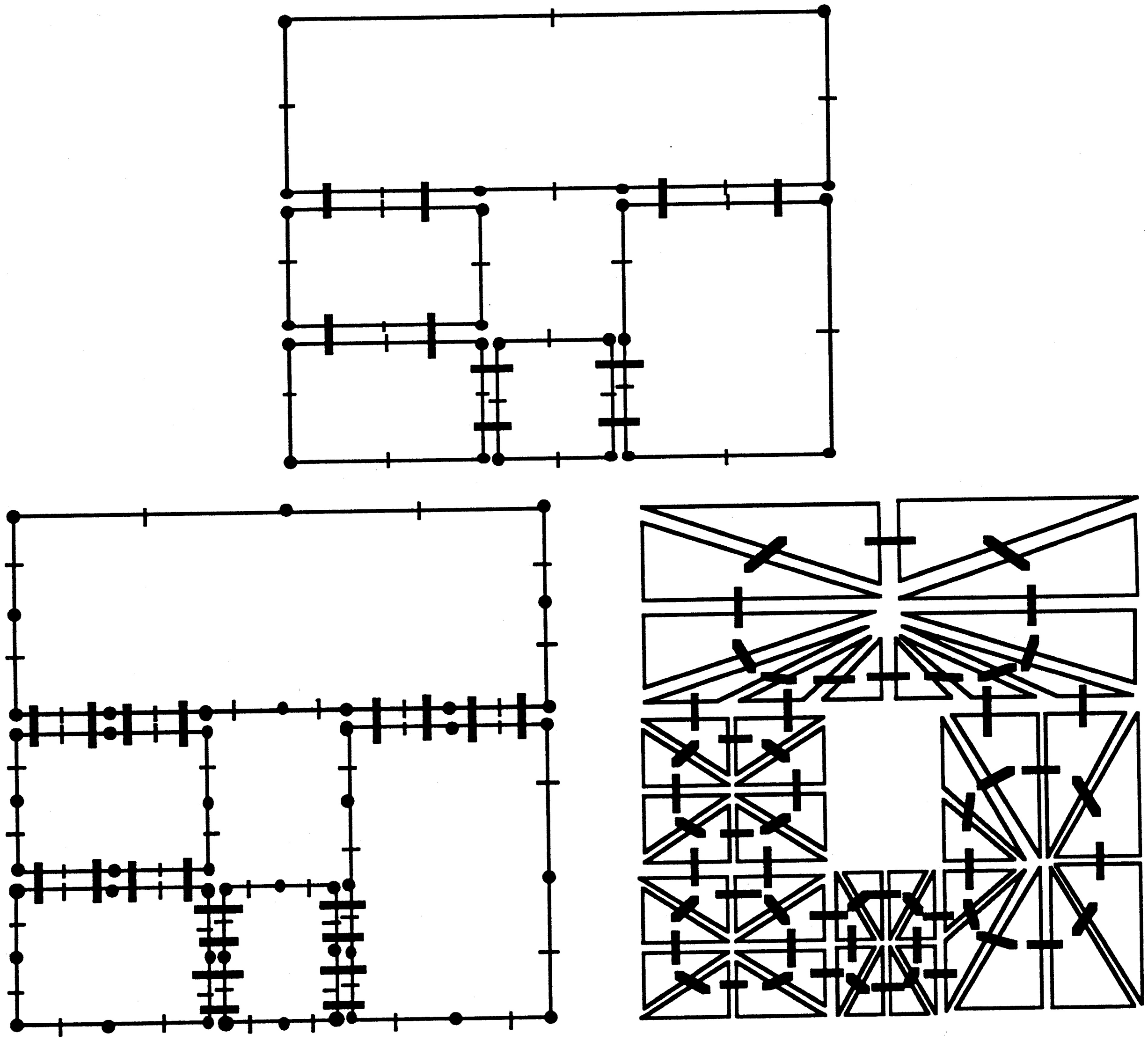


Fig. 3 : Up, a 2-G-map G is represented. Down and left, the barycentric 1-triangulation of G. Down and right, the barycentric 2-triangulation G^T of G (there is a 1-1 correspondance between the darts of G and the triangular faces of G^T ; in order to simplify the drawing of G^T, darts b and $b\alpha^T{}_0$ are not distinguished).