Good Triangulations in Plane*

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1 Introduction

Triangulation of a point set or a region in the plane is a very important problem since it has a number of applications in several areas. It requires that any two triangles either do not meet or meet at a vertex or at a full side. Though a lot of literature is available today on the topic, very few of them are addressed to the problem of guaranteed quality triangulation where the triangles are guaranteed to have some desirable qualities such as good bounds on their maximum and minimum angles. In fact, in finite element method, it is often desirable that the triangles have no obtuse angles. See [2]. Such triangles are called nonobtuse triangles. A triangulation which have only nonobtuse triangles is called a nonobtuse triangulation. It is known that if a given set of points admits a nonobtuse triangulation, the delaunay triangulation of the point set is nonobtuse. See [4].

In finite element method, the error bound is kept low if the triangles are as close as equilateral triangles. In [1] Babuska et. al showed that in finite element approximations with triangular elements, the smaller the maximum angle is, the lower the error bound becomes. Small angles are also not desirable since they yield ill-conditioned matrices [5]. In [2] Baker et. al have given a method for nonobtuse triangulation of a polygonal region. Their algorithm provides a tedious method for triangulating the regions near the boundary and do not work if the triangulation needs to include some prespecified input points inside the polygon, a requirement that often arises in interpolation techniques used for geological data. In [3], Chew has given an algorithm based on delaunay triangulation which triangulates a planar region and produces triangles with some guaranteed qualities.

Results: We give an algorithm for triangulating a planar point set which may or may not be given inside a polygonal boundary. If the point set is not given inside a polygon, we consider the convex hull of the given point set as the polygon containing it. In Our first algorithm given in Section 3, we follow the approach of [3]. It produces triangles with the following ensured qualities. (i) All obtuse triangles have angles between 30° and 120° and have sides of length in between d and $2d$ for some d. (ii) All triangles with a boundary edge as one of the sides, have angles in between 38.9° and 97.2° and have sides of length in between d and $1.5d$.

The second algorithm given in Section 4 produces a triangulation in which all obtuse triangles have angles in between 12° and 101°. This algorithm is much more simpler than the one given in [2] and achieves reasonably good bounds on the angles. Moreover, it works not only for triangulating a polygon but also a point set given inside a polygon.

2 Definitions and Geometric Lemmas

For any triangulation of a set of points, we call those edges as the boundary edges which have only one triangle incident on them. For a given set S of points in a polygonal region, the triangulation of S includes the edges of the given polygon. These edges appear as boundary edges of the triangulation. If the set S is not given inside a polygon, the boundary edges of the triangulation of S are the edges of the convex hull of the given point set.

A triangle in any triangulation is called a boundary triangle if it has a boundary edge as its side. The triangles which are not boundary triangles are called inner triangles.

A triangle in any triangulation T is said to have good circumcenter if and only if the line segments joining the circumcenter and three vertices of the triangle do not intersect any boundary edge of T. Conversely, a triangle is said to have bad circumcenter if and only if one of the line segments joining the circumcenter to its vertices intersect a boundary edge of T.

The following lemmas are used in the next section. Proofs appear in the full paper.

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Lemma 2.1: Let $\overline{p_ip_j}$ and $\overline{p_xp_y}$ be two chords drawn in a semicircle U. Further, let $\overline{p_xp_y}$ lie in between $\overline{p_ip_j}$ and a diameter of U. Then, $|\overline{p_xp_y}| \geq |\overline{p_ip_j}|$

Lemma 2.2: Let $\overline{p_ip_j}$ be a chord in a circle C and p_m be a point which lies outside C and on the same side of $\overline{p_ip_j}$ as the center of C does. Further, let $\angle p_mp_ip_j$ and $\angle p_ip_jp_m$ be nonobtuse. Then center of C lies inside $\Delta p_ip_jp_m$.

Lemma 2.3: Let $G = \Delta p_ip_jp_k$ be an obtuse triangle with bad circumcenter in any delaunay triangulation. Let $\overline{p_ip_k}$ be the largest side of $\Delta p_ip_jp_k$. There must exist a boundary edge which is greater than or equal to $\overline{p_ip_k}$.

3 Using Delaunay Triangulation

Algorithm: Let S be a set of points given in the plane. In what follows, by delaunay triangulation of a point set S inside a polygon we mean the constrained delaunay triangulation of S [6]. Algorithm $TRI1$ as given below produces a guaranteed quality triangulation.

Two input conditions must be satisfied for the algorithm $TRI1$. Later, we will see how these conditions are met.

Algorithm TRI1:

Input Conditions: Let d be a quantity, such that no two given points are closer than d and no boundary edge is greater than $1.5d$ and less than d.

begin

Construct the delaunay triangulation of the given set of points.

Repeat

Add the circumcenter v_l of $G = \Delta p_ip_jp_k$ satisfying following properties.

1. G is obtuse

2. v_l is at a distance of at least d from all the three points p_i, p_j, p_k.

3. v_l is a good circumcenter of $\Delta p_ip_jp_k$.

Update the current triangulation.

Until there is no such circumcenter.

end

The proofs of some of the following lemmas are omitted here. They appear in the full paper.

Lemma 3.1: Algorithm $TRI1$ terminates.

Proof: By the empty circle property of the delaunay triangulation, the added points are at a distance of at least d from all other points. Since, there can be finitely many such points in a bounded region, algorithm $TRI1$ can add finite number of new points and thus terminates. ♣.

Lemma 3.2: Each obtuse triangle G with good circumcenter produced by $TRI1$ have the following criteria. G has no angles greater than $120°$ and less than $30°$ and moreover, G has no edge greater than $2d$ and less than d.

Certainly, circumradius of G is less than d. Otherwise, its circumcenter would have been introduced by $TRI1$. The sides of G are of length greater than or equal to d. It is easy to prove that a triangle with all its sides greater than its circumradius must have angles in between $30°$ and $120°$.

Lemma 3.3: Let $G = \Delta p_ip_jp_k$ be a triangle produced by $TRI1$ which satisfies the following conditions.

1. G is obtuse with the obtuse angle $\angle p_ip_jp_k$.

2. G has good circumcenter.

3. $|\overline{p_ip_j}| \geq \sqrt{2}d$ and $|\overline{p_jp_k}| \geq d$.

Let $\overline{p_jp_l}$ intersect $\overline{p_ip_k}$ inside at p_l and $\angle p_ip_jp_l$ be obtuse as shown in Figure 3.1. Then, $|\overline{p_ip_l}| > 1.58d$.

Lemma 3.4: Any triangle G produced by $TRI1$ which has bad circumcenter is a boundary triangle. Moreover, G has a boundary edge as the opposite side of the obtuse angle.

Lemma 3.3 is used to prove this Lemma.

Lemma 3.5: Each inner obtuse triangle G produced by $TRI1$ have angles in between $30°$ and $120°$ and sides of length in between d and $2d$.

Proof: Combine Lemma 3.2 and Lemma 3.4.♣

Lemma 3.6: Let $G = \Delta p_ip_jp_k$ be a triangle such that each of its sides has length between d and $1.5d$. Then, the angles of $\Delta p_ip_jp_k$ must be in between $38.9°$ and $97.2°$.

Theorem 3.1: Triangles produced by the algorithm $TRI1$ satisfies the following conditions.(i) Each inner obtuse triangle has angles in between $30°$ and $120°$ and sides of length in between d and $2d$. (ii) Each boundary obtuse triangle has angles in between $38.9°$ and $97.2°$ and sides of length in between d and $1.5d$.

Proof: According to Lemma 3.4 each obtuse boundary triangle has a boundary edge as the opposite side of the obtuse angle. Since each boundary edge has length in between d and $1.5d$, these triangles must have sides of length in between d and $1.5d$. Further, according to Lemma 3.6 these triangles have angles in between $38.9°$ and $97.2°$. By Lemma 3.5, each inner obtuse triangle produced by $TRI1$ satisfies the stated conditions.♣

Input Conditions of TRI1: Let δ_1 be the minimum distance between any two points. Let δ_2 be the minimum distance between a point and a boundary edge and δ_3 be the minimum length of any boundary edge. Let $d = min(\delta_1, \delta_2, \frac{\delta_3}{3})$. Definitely, each boundary edge is greater than or equal to $3d$. It is easy to divide such edges into segments which have lengths in between d and $1.5d$. This introduces new points which can not be closer than d to any other points by the choice of d. Again, no two points can be closer than d. Thus, d satisfies all the input conditions of the algorithm $TRI1$.

Complexity of TRI1: The time complexity of the algorithm depends on the time of updation. Clearly, each updation can be done in $O(n)$ time where n is the number of points present in the output. This gives a time complexity of $O(n^2)$, though on the average, the number of points affected by each update remains constant and thus the algorithm runs in $O(n)$ time on the average.

Let the area of the boundary to be triangulated be A. Since the distance between any two points is at least d in the output produced by the algorithm, the number of points added is bounded by the number of equilateral triangles with sides of length d which can fit in the area A. Thus, $TRI1$ produces at most $\frac{4A}{\sqrt{3d^2}}$ triangles.

4 Triangulation with Grid

In [2], Baker et. al have given an algorithm to triangulate a simple polygon with nonobtuse triangles. They overlay a square grid on the polygon and observe that each inner square through which no boundary edge passes, can be triangulated with two right angled triangles. The difficult part is to triangulate the squares through which a boundary edge passes. See Figure 4.1. They give a tedious method to triangulate these regions into nonobtuse triangles. Here, we give a very simple method to triangulate these regions so that all obtuse triangles have angles between 12° and 101°. Our algorithm allows input points inside and on the boundary of the given polygon.

Lemma 4.1: Let $\overline{p_ip_j}$ be a line segment. Let $p_i\vec{p}_s$ and $p_j\vec{p}_t$ be two rays, perpendicular to $\overline{p_ip_j}$ at p_i and p_j. Further, let $\overline{p_lp_m}$ be another line segment, parallel to $\overline{p_ip_j}$ intersecting $p_i\vec{p}_s$ and $p_j\vec{p}_t$ at p_l and p_m respectively. Let p_k be a point which lie in the shaded region as shown in Figure 4.2. The angle $\angle p_jp_kp_i$ is maximized when p_k is the midpoint of the line segment $\overline{p_lp_m}$.

Lemma 4.2: Let $\overline{p_ip_j}$ be a line segment. $p_i\vec{p}_s$ and $p_j\vec{p}_t$ are two rays perpendicular to $\overline{p_ip_j}$ drawn at p_i and p_j respectively. Let p_l, p_m and p_k be three points on $p_i\vec{p}_s, p_j\vec{p}_t$ and $\overline{p_ip_j}$ respectively as shown in Figure 4.3. Further, let $|\overline{p_ip_l}| = S$, $|\overline{p_jp_m}| = L$ and $|\overline{p_ip_j}| = T$. Let S_{min}, L_{min} denote minimum values of S and L respectively and T_{max} denote the maximum value of T. The maximum value of $\angle p_mp_kp_l$ is $180 - tan^{-1}(\frac{S_{min}}{x_0}) - tan^{-1}(\frac{L_{min}}{T_{max}-x_0})$ where

$$x_0 = \frac{\sqrt{T_{max}^2S_{min}^2 + (L_{min} - S_{min})(T_{max}^2S_{min} + L_{min}S_{min}(L_{min} - S_{min}))} - T_{max}S_{min}}{(L_{min} - S_{min})}$$

Lemma 4.3: Let S be a set of points in the simple polygon P containing no acute interior angles. S includes the points corresponding to the vertices of the polygon. By introducing points inside P and on the boundary of P, S can be triangulated in such a way that each obtuse triangle has angles between 12° and 101°.

Proof: We draw horizontal and vertical lines through each point in S. This forms a rectangular grid. We refine the grid so that through any rectangle no two nonadjacent boundary edges pass and each side of any rectangle has length in between d and $1.5d$ for some d. The choice of d and the method of this refinement are discussed later. We introduce the gridpoints which are inside P and also the points where the gridlines intersect the boundary. We introduce the edges between these points which are on the grid. Each internal rectangle through which no edge passes, can be triangulated into two right angled triangles by a diagonal. While triangulating the rectangles through which a boundary edge passes, we introduce points only inside P or on the boundary of P, but not on the sides of the rectangles. Thus, each rectangle can be triangulated independently without propagating points to the adjacent rectangles.

If two boundary segments pass through a rectangle, they must be adjacent. Since the interior angles between the corresponding boundary edges is obtuse, the regions of P bounded by these two boundary segments and the sides of the rectangle must be disjoint. Thus, we can triangulate these two regions independently. This implies that we need to worry about how to triangulate the region in P bounded by a boundary segment and sides of a rectangle without introducing points on the sides of the rectangle except at the corner points and the points where boundary intersects them.

Let $abcd$ be a rectangle through which the boundary segment $\overline{pq}$ passes. W.l.o.g., assume b to lie inside P.

Case(i): See Figure 4.4. The triangle is a nonobtuse triangle.

Case(ii): See Figure 4.5. In this case carry out the triangulation as shown. The angle $\angle pqb$ is nonobtuse since q lies outside the circle drawn with the diameter $\overline{ab}$. This is because of the fact that the maximum length of $\overline{ab}$ is $1.5d$ and the minimum length of $\overline{cd}$ is d.

Case(iii): See Figure 4.6. Let $|\overline{ab}| = T$ and $|\overline{ad}| = L$. Without loss of generality, assume $T \geq L$. Draw a line segment $\overline{st}$ which is parallel to $\overline{ab}$ and at a distance of $\frac{T}{2}tan40°$ from it. We have two subcases depending on the position of p. Consider the case where p lies on $\overline{as}$. If $\angle qpb$ is obtuse, carry out the triangulation as shown in Figure 4.6(a). Let

$\overline{qu}$ be the perpendicular line to $\overline{cd}$ at q and $\overline{pu}$ be the perpendicular line to $\overline{pb}$ at p. These two lines meet at u. Join u to b and c. This may render $\angle pub$ to be obtuse which can be resolved by dropping a perpendicular from u on $\overline{pq}$. We prove that the angle $\angle cub$ can never be obtuse. As can be seen $\angle qpb = 90° + \beta - \alpha$. For $\angle qpb$ to be obtuse, we must have $\beta > \alpha$ which implies $|\overline{dq}| < \frac{|\overline{ap}||\overline{pd}|}{|\overline{ab}|}$ Let $|\overline{ap}| = x$. We have $|\overline{dq}| < \frac{x(L-x)}{T}$. This gives $|\overline{dq}| < \frac{L^2}{4T} \leq \frac{T^2}{4T} = \frac{T}{4}$. Thus maximum value of $|\overline{dq}|$ is $0.375d$. This immediately implies u lies outside the semicircle drawn with the diameter $\overline{bc}$. Hence, $\angle cub$ must be nonobtuse.

Consider the case when instead of $\angle qpb$, the angle $\angle bqp$ is obtuse. We know the minimum values of $|\overline{bc}|$ and $|\overline{pd}|$ are d and $(d - \frac{1.5d\ tan40°}{2})$ respectively and the maximum value of $|\overline{dc}|$ is $1.5d$. Applying Lemma 4.2 with these values we get the maximum value of $\angle bqp$ to be less than 101°. It can be proved that minimum values of $\angle qbp$ and $\angle qpb$ are greater than 12°.

Let us consider the subcase when p lies on $\overline{ds}$. If the angle $\angle qpb$ is obtuse, one can obtain nonobtuse triangulation in the same way as discussed in the previous subcase. If instead, $\angle bqp$ is obtuse, carry out the triangulation as in the case when $\angle qpb$ is obtuse, but with the role of p and q switched. See Figure 4.6(c). In this case the only angle which may be obtuse is $\angle bua$. Since $|\overline{as}| = \frac{|\overline{ab}|tan40°}{2}$, the angle $\angle bua$ is 100° when u is the midpoint of $\overline{st}$. Thus, by Lemma 4.1 $\angle bua$ has the maximum value of 100°. It easy to see that the minimum value of angles $\angle uab$ and $\angle abu$ is greater than $tan^{-1}(\frac{tan40°}{2})$ which is greater than 22°.

This exhausts all possible cases and we observe that all obtuse triangles produced by this method have angles between 12° and 101°.♣

Generating grids with proper spacings: Draw a horizontal and a vertical line through each point to have an initial grid and then refine this grid so that no two nonadjacent boundary segments pass through a rectangle. Let h be the minimum spacing between any two adjacent grid lines. Take $d = \frac{h}{3}$. Definitely, with this choice of d, two adjacent gridlines have spacing of at least $3d$. It is easy to refine such a grid so that every adjacent gridline spacing lies between d and $1.5d$.

Theorem 4.1: Let S be a set of points in the simple polygon P . S includes the points corresponding to the vertices of the polygon. By introducing points inside P and on the boundary of P, S can be triangulated in such a way that each obtuse triangle has angles between 101° and 12°.

Proof: As discussed in [2], corresponding to each vertex of P where the interor angle is acute, we can cut off a triangular portion, such that the cut off triangle is nonobtuse and does not contain any point inside. After this modification, we apply Lemma 4.3 on the new polygon thus generated from P which does not have any acute interior triangle. This may introduce points on the side of the cut off triangles which is not a boundary segment. Such triangles with those added points can be triangulated with nonobtuse triangles as shown in Figure 4.7.♣

The algorithm based on voronoi triangulation can easily be extended to 3-D which guarantees the face angles of each tetrahedron to be in between $\angle 30°$ and $\angle 120°$. But, this does not ensure any good bounds on the dihedral angles or solid angles of the tetrahedra. Currently, research is going on to find out algorithms for good triangulations in 3-D.

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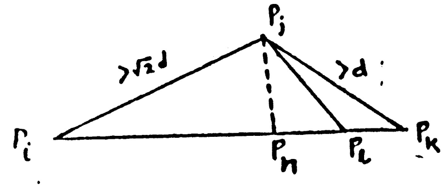


Figure 3.1

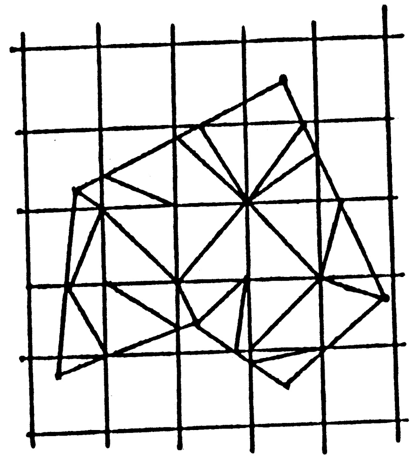


Figure 4.1

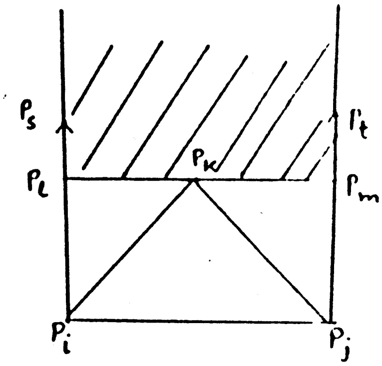
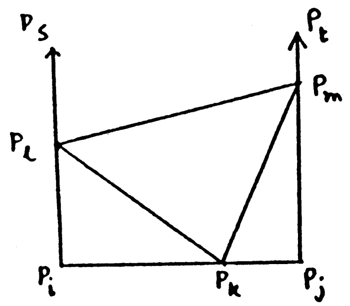


Figure 4.2



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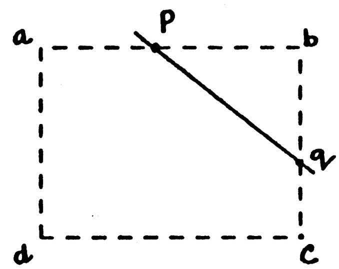

Figure 4.4

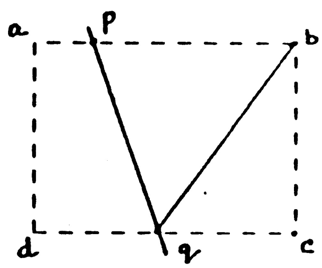

Figure 4.5

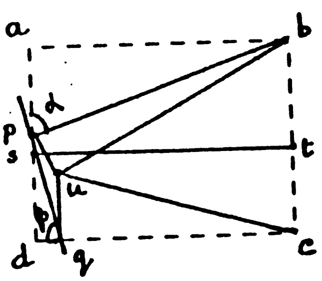

Figure 4.6(a)

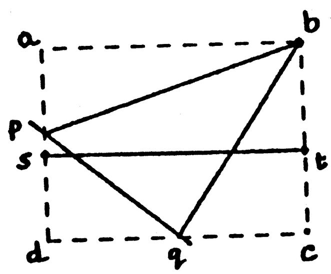

Figure 4.6(b)

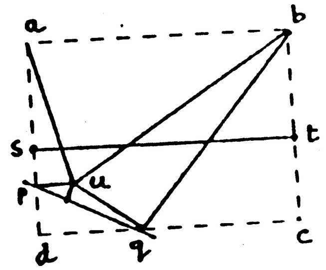

Figure 4.6(c)

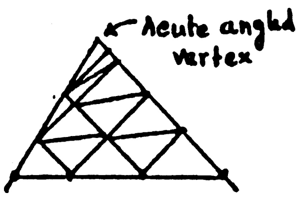

Figure 4.7