Improved Combinatorial Bounds and Efficient Techniques for Certain Motion Planning Problems with Three Degrees of Freedom

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1 Introduction

We study motion planning problems for several systems with three degrees of freedom. These problems can be rephrased as the problem of analyzing the combinatorial complexity of a single cell in arrangements of certain types of surfaces (actually, surface patches) in 3-dimensional space. The combinatorial complexity of the entire arrangement (or even only its portions that represent free placements of the robot) in each case that we study can be $\Theta(n^3)$ in the worst case. Nevertheless, for each such arrangement we obtain a subcubic bound on the total combinatorial complexity of all the 3D cells in the arrangement that contain a portion of the 1D boundary of a surface patch in their closure (these are called the *interesting* cells); the bound is $O(n^{7/3})$ in the case of arrangements related to the motion planning problem of a so-called telescopic arm moving in the plane among polygonal obstacles with n corners, and $O(n^{5/2})$ in the case of arrangements resulting from the motion planning problem for an L-shaped object in the plane amidst n point obstacles. Each non-interesting cell is shown to have much smaller complexity ($O(n)$). We also devise an algorithm to compute the interesting cells in the second type of arrangements, whose time complexity is $O(n^{5/2}\log^2 n)$, and an algorithm with running time $O(n^{7/3})$ for the case of a telescopic arm moving among point obstacles, in both cases improving over the best previously known algorithms for these problems, whose time complexity is $O(n^3 \log n)$. Thus our results lead to subcubic solutions of the motion planning problems that induce these arrangements, since it suffices to calculate only the cell of the arrangement that contains the initial placement of the robot.

Our approach reduces each three-dimensional problem into a collection of problems involving two-dimensional arrangements. To solve these two-dimensional problems we obtain two combinatorial results of independent interest for arrangements in the plane: (i) a tight bound $\Theta(nm^{1/2})$ on the maximum joint combinatorial complexity of m "concave chains" in an arrangement of n pseudo lines, and (ii) an upper bound $O(m^{2/3}n^{2/3} + n\alpha(n))$ on the maximum number of edges of m distinct faces in certain types of arrangements of n pseudo segments, which is within an $\alpha(\cdot)$ factor off the lower bound for this quantity.

We regard our results as significant, because: (i) They have "widened the crack at the door" opened in [ArS] concerning the complexity of a single cell in an arrangement of surfaces in 3-space, obtaining subcubic bounds for cases resulting from motion planning problems involving *rotation*, where the surfaces have a much more complex shape. Until the major (and extremely difficult) conjecture in this area is settled, namely that a single cell in such an arrangement has only near-quadratic complexity, we would need to continue to extend this type of analysis to progressively more complex kinds of surfaces (and motion planning instances). We regard the current work as a significant step in this direction. (ii) They have led to independent and interesting results concerning arrangements of pseudo lines and pseudo segments in the plane, exemplifying yet another time the strong interaction between 2D and 3D arrangements.

Many details as well as all the proofs were omitted in this short summary. They all appear in the full version of the paper ([HS]).

[0] Work on this paper has been supported in part by the U.S.-Israeli Binational Science Foundation, the NCRD - the Israeli National Council for Research and Development, and the Foundation for Research in Electronics, Computers and Communications, administered by the Israeli Academy of Sciences and Humanities. Work by the second author has also been supported by Office of Naval Research Grants N00014-87-K-0129, N00014-89-J-3042 and N00014-90-J-1284, and by National Science Foundation Grant CCR-89-01484.

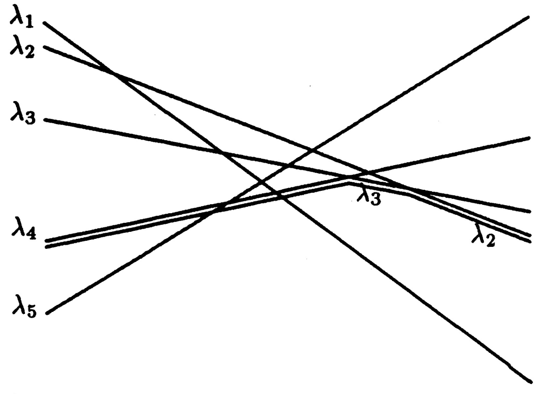

Figure 1: A concave chain

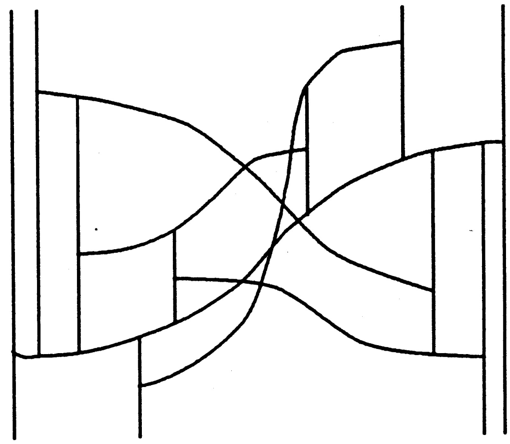

Figure 2: Vertically partitioned arrangement of pseudo segments

2 Combinatorial results for two-dimensional arrangements

Since the motion planning problems that we study involve a rotational degree of freedom, the resulting surfaces and hence the three-dimensional arrangements are rather convolute and hard to visualize and to analyze directly. We overcome this difficulty by reducing each three-dimensional problem into a collection of problems involving two-dimensional arrangements. In this section we summarize our combinatorial results for arrangements in the plane that are needed to solve these two-dimensional problems.

2.1 Concave chains in arrangements of pseudo lines

Let $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ be a collection of n *pseudo lines* in the plane, that is x-monotone unbounded curves, each pair of which intersect at most once. We order Λ by "slope", namely by the reverse vertical order of the pseudo lines at $x = -\infty$ (so $\lambda < \lambda'$ if λ lies higher than λ' at $x = -\infty$). This is a linear order. Let us denote by $\mathcal{B} = \mathcal{B}(\Lambda)$ the *arrangement* of the pseudo lines in Λ.

A *concave chain* c in $\mathcal{B}$ is an x-monotone (connected) path that is contained in the union of the pseudo lines of Λ, intersects every vertical line (once), and the sequence of pseudo lines traversed by c from left to right is a strictly decreasing sequence. (See Figure 1 for an illustration.)

Informally, as we traverse c from left to right, whenever we reach a vertex of $\mathcal{B}$, we can either continue along the pseudo line we are currently on, or make a right (i.e. downward) turn onto the other pseudo line (having a smaller slope), but we cannot make a left (upward) turn; in case the pseudo lines are real lines, c is indeed a concave polygonal chain. It is clear that the number of turns along a concave chain is at most $n - 1$.

We study the following problem: Given m concave chains in $\mathcal{B}$, what is the maximum *joint combinatorial complexity* of these chains, defined as the number of vertices of $\mathcal{B}$ at which at least one chain makes a turn. Note that such a vertex can be shared by many chains, which overlap near that turn, but we count it only once. Note also that we do not count a vertex in which two chains cross each other (without turning), unless a third chain does make a turn there. Our result is:

Lemma 2.1 *The maximum joint combinatorial complexity of m concave chains in an arrangement of n pseudo lines is $\Theta(n\sqrt{m})$.*

We also need a variant of this result which states:

Lemma 2.2 *The maximum joint combinatorial complexity of m concave chains in an arrangement of n pseudo lines where each turn is counted as many times as there are chains making it and where at most r chains make the same turn, is $\Theta(n\sqrt{mr})$.*

2.2 Many faces in certain arrangements of pseudo segments

We wish to analyze the maximum combinatorial complexity, $K(m, n)$, of m faces in an arrangement $\mathcal{B}'$

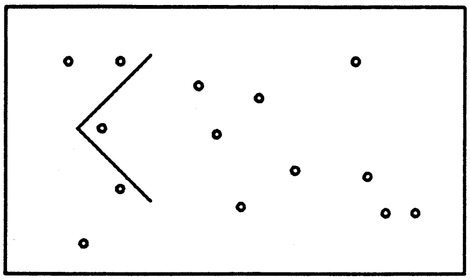

Figure 3: An L-shaped object

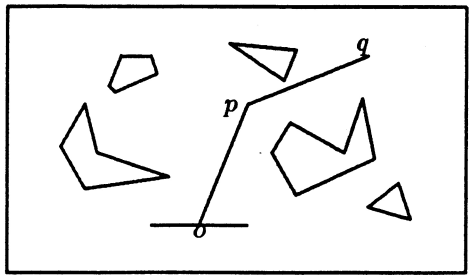

Figure 4: A telescopic arm

of n pseudo segments in the plane, where the pseudo segments are x-monotone, they are contained in a collection of n pseudo lines, and the faces of the arrangement are also partitioned by vertical straight line segments through the endpoints of all the pseudo segments that stretch until they hit another pseudo segment (Figure 2).

We have the following result, using the technique of [CEGSW] (A similar result has been independently obtained by Boris Aronov.)

Lemma 2.3 $K(m,n) = O(m^{2/3}n^{2/3} + n\alpha(n))$.

3 The arrangement of an L-shaped object

We consider the motion planning problem for an L-shaped object moving in the plane amidst n point obstacles, whose study has been initiated in [HOS]; see Figure 3. Each placement of the object can be parametrized by (x, y, θ) where (x, y) is the position of the internal corner of the object, and θ is its orientation. The problem can be transformed to the analysis of an arrangement $\mathcal{A}$ of n "contact surfaces" in $R^2 \times S^1$, which decomposes the 3-dimensional space into pairwise disjoint connected cells, each of 0,1,2 or 3 dimensions. We shall use the unquantified term *cell* for a 3-dimensional cell of $\mathcal{A}$. A cell c of $\mathcal{A}$ is *interesting* if its closure $\bar{c}$ contains a portion of the boundary of some surface (patch). All other cells of the arrangement are called *dull* (this terminology is borrowed from [ArS]). We analyze the total combinatorial complexity of all the interesting cells in the arrangement $\mathcal{A}$, i.e., the number of faces, edges, and vertices of all these cells. It is shown in [HOS] that the worst-case total combinatorial complexity of the entire $\mathcal{A}$ is $\Theta(n^3)$. One of our goals is to bound the combinatorial complexity of a single cell in $\mathcal{A}$. Our analysis provides a subcubic upper bound on this complexity, but the actual maximum complexity of a single cell might be much smaller than that, and this problem still needs to be studied. We also remark that a single interesting cell can easily be shown to have $\Omega(n^2)$ combinatorial complexity in the worst case.

Using, among other things, Lemma 2.2 for concave chains, we obtain

Theorem 3.1 *The total combinatorial complexity of the interesting cells in the arrangement $\mathcal{A}$ induced by an L-shaped object moving in the plane amidst n point obstacles, is $O(n^{5/2})$.*

4 The arrangement of a telescopic arm

Aronov and Ó'Dúnlaing ([AO]) consider the following planar robot arm. It consists of two links, $\overline{op}$ and $\overline{pq}$. o is an anchor point. The first link $\overline{op}$ is a telescopic link which can rotate around o, and extend or shrink along its length. The second link $\overline{pq}$ has a fixed length d, and can rotate around p; see Figure 4. In [AO] this robot arm is called a *telescopic arm*. It is shown in [AO] that the configuration space of this arm moving among polygonal obstacles has $\Theta(n^3)$ connected components in the worst case, and an $O(n^3 \log n)$-time and $O(n^3)$-space algorithm is presented to compute it.

We analyze the combinatorial structure of the arrangement induced by the motion of such an arm among polygonal obstacles having a total of n corners. The terms *cell* and *interesting cell* are defined as in the previous section for the L-arrangement.

Using Lemma 2.3 and additional tools we obtain

Theorem 4.1 *The total combinatorial complexity of all the interesting cells in the arrangement induced* *by the motion planning for a telescopic arm among polygonal obstacles is* $O(n^{7/3})$.

5 Computing the interesting cells and find-path algorithms

We distinguish between the *reachability* problem which is to decide whether a collision-avoiding path between the initial placement of the moving object and the desired final placement exists, and the *find-path* problem which is to actually compute the path if it exists.

We combine the combinatorial result of Section 3 with the decision algorithm of [HOS] to obtain an efficient algorithm for solving the find-path problem for an L-shaped object which by a slight modification can be made into an algorithm for computing the interesting cells of an L-arrangement. The algorithm runs in $O(n^{5/2}\log^2 n)$ time and requires $O(n^{5/2})$ space.

Similarly we obtain an algorithm for solving the find-path problem for a telescopic arm moving among point obstacles. The algorithm requires $O(n^{7/3})$ time and space.

References

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