Characterizing weak visibility polygons and related problems

Subir Kumar Ghosh* Anil Maheshwari† Sudebkumar Prasant Pal‡
Sanjeev Saluja‡ C. E. Veni Madhavan‡

Abstract

In this paper we characterize a weak visibility polygon in terms of the shortest path between two arbitrary vertices of the polygon. This characterization establishes the relationship between convexity and visibility and more precisely, the characterization captures the convexity of shortest paths in a weak visibility polygon. Due to convexity of shortest paths, the characterization leads to three efficient algorithms for weak visibility polygons as follows. We propose a linear time algorithm to determine whether a polygon P is weakly visible from a given segment st. This algorithm can be viewed as a simplification of the linear time algorithm of Avis and Toussaint [AT81] for the same problem. If P is weakly visible from st, we propose a linear time algorithm for computing the shortest path tree rooted at any vertex of P. The recent result of Heffernan and Mitchell [HM89] can also solve the same problem in linear time using an entirely different technique. If the given polygon P of n vertices is weakly visible from a convex edge, we propose an $O(n^2)$ time algorithm for computing the maximum hidden set in P. The problem of computing the maximum hidden set in an arbitrary polygon is known to be NP-hard and no polynomial time algorithms are known for any special class of polygons.

1. Introduction

In the recent past, shortest paths have been used to compute the weak visibility polygon of a polygon P from a given segment [GHLST87,T86] or a set [G86] inside P. If a polygon is weakly visible from an internal segment st then it is known that for any vertex v, the shortest path from s to v, the shortest path from t to v and the segment st form a funnel, where st is the base of the funnel and v is the cusp of the funnel. However, no property of the shortest path between two arbitrary vertices of a weak visibility polygon is known. Here we characterize weak visibility polygons in terms of the shortest path between two arbitrary vertices. The characterization establishes the relationship between convexity and visibility and more precisely, the characterization captures the convexity of shortest paths in a weak visibility polygon. Due to convexity of shortest paths the characterization leads to three efficient algorithms as follows. We propose a linear time algorithm to determine whether a polygon P is weakly visible from a given internal segment st (omitted in this version, see [GMPSV90]). This algorithm can be viewed as a simplification of

*Computer Science Group, Tata Institute of Fundamental Research, Bombay-400005, INDIA

†Computer Systems and Communications Group, Tata Institute of Fundamental Research, Bombay-400005, INDIA

‡Department of Computer Science and Automation, Indian Institute of Science, Bangalore-560012, INDIA

the algorithm of Avis and Toussaint [AT81] for the same problem. If P is weakly visible from st, we propose a linear time algorithm for computing the shortest path tree rooted at any vertex of P and the doubly linked list is the only data structure required in the algorithm (omitted in this version, see [GMPSV90]). The recent result of Heffernan and Mitchell [HM89] can also solve the same problem in linear time using an entirely different technique. However, it is also possible in linear time to compute the shortest path tree in a weak visibility polygon P where the segment st is not given (see [PGV90]). Our algorithm for computing the shortest path tree and the algorithm in [PGV90] use two different techniques though both algorithms are based on the characterization of weak visibility polygons. If the given polygon P is not weakly visible from any internal segment, the best known algorithm for computing the shortest path tree runs in $O(n)$ time [GHLST87, Ch90]. If the given polygon P of n vertices is weakly visible from a convex edge, we propose an $O(n^2)$ time algorithm for computing the maximum hidden set in P. The maximum hidden set of a polygon is the maximum cardinality set of vertices such that no two vertices of the set are mutually visible. The problem of finding the maximum hidden set in an arbitrary polygon is known to be NP-hard [S87] and no polynomial time algorithms are known for any special class of polygons.

We assume that the simple polygon P is given as a counterclockwise sequence of ***vertices*** $v_1, v_2, \ldots, v_n$ with their respective x and y coordinates. We assume that no three vertices of P are collinear. The line segments $v_1v_2, \ldots, v_{n-1}v_n, v_nv_1$ are called *edges* of P. The symbol P is used to denote the region of the plane enclosed by P and $bd(P)$ denotes the boundary of P. If p and q are two points on $bd(P)$ then the counterclockwise $bd(P)$ from p to q is denoted as $bd(p,q)$. If the interior angle at the vertex v_i is convex, then v_i is called a ***convex*** vertex. An edge v_iv_{i+1} of P is called a ***convex*** edge if both v_i and v_{i+1} are convex vertices. Two points are said to be ***visible*** if the line segment joining them lies totally inside P. If the line segment joining two points touches $bd(P)$, they are still considered to be visible. A point p is said to be ***weakly visible*** from an edge or an internal segment st, if there is a point z in the ***interior*** of st such that p and z are visible. If every point in P is weakly visible from st then P is said to be ***weakly visible from st*** and P is called a ***weak visibility polygon***. If a polygon P is weakly visible from a convex edge v_kv_{k+1}, we call the edge v_kv_{k+1} a ***convex visibility edge***. Let $SP(u,v)$ denote the Euclidean shortest path inside P from a point u to another point v. For any vertex u of P the ***shortest path tree of P rooted at u***, denoted as $SPT(u)$, is the union of the shortest paths from u to all the vertices of P. Let st be an internal segment of P such that s and t belong to $bd(P)$. If st cannot be extended either from s or from t without intersecting the exterior of P, we call it a ***chord*** of P. If a polygon P is weakly visible from st, st is called a ***visibility chord*** of P. A chord st of P divides P into two polygons and we call them the ***subpolygons of st***.

The paper is organized as follows. In Section 2 we characterize polygons weakly visible from a convex edge or a chord. In Section 3 we propose an algorithm for computing the maximum hidden set in a polygon weakly visible from a convex edge.

2. Characterizing weak visibility polygons

In this section, we characterize polygons weakly visible from a convex edge or a chord in terms of shortest paths. We start by stating the known properties of shortest paths in weak visibility polygons. If a vertex v_i of P is weakly visible from a convex edge v_kv_{k+1} of P, then the following properties hold.

1. $SP(v_{k+1}, v_i)$ makes a right turn at every vertex in the path.

2. $SP(v_k, v_i)$ makes a left turn at every vertex in the path.

3. $SP(v_k, v_i)$ and $SP(v_{k+1}, v_i)$ are two disjoint paths and they meet only at v_i.

4. The region enclosed by $SP(v_k, v_i)$, $SP(v_{k+1}, v_i)$ and $v_k v_{k+1}$ is totally contained inside P.

It can be seen that the above properties are of shortest paths between the vertices v_k and v_{k+1}, and any other vertex v_i of P. So, they do not suggest any property of the shortest path between two arbitrary vertices v_i and v_j of P. However, the shortest path between two vertices of a weak visibility polygon do satisfy a property as shown in the following lemmas. We omit the proofs of all the lemmas in this version.

Before we state the lemmas, we need some notations. For any two vertices v_i and v_j, ***chain***(v_i, v_j) denotes either $bd(v_i, v_j)$ or $bd(v_j, v_i)$. If *chain*(v_i, v_j) contains an edge $v_k v_{k+1}$, we call it ***same chain*** $(v_i, v_j, v_k v_{k+1})$. If *chain*(v_i, v_j) does not contain $v_k v_{k+1}$, we call it ***opposite chain***$(v_i, v_j, v_k v_{k+1})$. In Figure 1 the vertices of *same chain*$(v_2, v_6, v_3 v_4)$ are v_2, v_3, v_4, v_5, v_6 and the vertices of ***opposite chain***$(v_2, v_6, v_3 v_4)$ are v_6, v_7, v_1, v_2. We now state our lemmas.

Lemma 1: *If P is weakly visible from a convex edge $v_k v_{k+1}$ of P, then for any two vertices v_i and v_j where v_i belongs to $bd(v_{k+1}, v_j)$, all vertices of $SP(v_i, v_j)$ belong to opposite chain $(v_i, v_j, v_k v_{k+1})$* (Figure 2).

Lemma 2: *Let $v_k v_{k+1}$ be a convex edge of a polygon P. For any two vertices v_i and v_j of P where v_i belongs to $bd(v_{k+1}, v_j)$, if all vertices of $SP(v_i, v_j)$ belong to opposite chain $(v_i, v_j, v_k v_{k+1})$, then $SP(v_i, v_j)$ makes a right turn at every vertex in the path and $SP(v_j, v_i)$ makes a left turn at every vertex in the path.*

Lemma 3: *Let $v_k v_{k+1}$ be a convex edge of a polygon P. For any vertex v_i of P if $SP(v_{k+1}, v_i)$ makes a right turn at every vertex in the path and $SP(v_k, v_i)$ makes a left turn at every vertex in the path, then P is weakly visible from $v_k v_{k+1}$.*

Theorem 1: *Let $v_k v_{k+1}$ be a convex edge of a polygon P. The following statements are equivalent.*

1. *P is weakly visible from $v_k v_{k+1}$.*

2. *For any two vertices v_i and v_j of P where v_i belongs to $bd(v_{k+1}, v_j)$, $SP(v_i, v_j)$ passes only through vertices of opposite chain$(v_i, v_j, v_k v_{k+1})$.*

3. *For any two vertices v_i and v_j of P, where v_i belongs to $bd(v_{k+1}, v_j)$, $SP(v_i, v_j)$ makes a right turn at every vertex in the path and $SP(v_j v_i)$ makes a left turn at every vertex in the path.*

4. *For any vertex v_i of P, $SP(v_{k+1}, v_i)$ makes a right turn at every vertex in the path and $SP(v_k, v_i)$ makes a left turn at every vertex in the path.*

Proof: (i) implies (ii) by Lemma 1, (ii) implies (iii) by Lemma 2. (iii) implies (iv) as a special case and (iv) implies (i) by Lemma 3. **Q.E.D.**

The above theorem characterizes the polygons that are weakly visible from a convex edge. Now we consider polygons that are weakly visible from a chord st of P. The chord st divides P into two subpolygons, one bounded by st and $bd(s, t)$ and the other by st and $bd(t, s)$. It can be seen that st is a convex edge of both the subpolygons and therefore, Theorem 1 holds for each of them.

So far we have stated the properties of the shortest path between two vertices of the same subpolygon of a visibility chord. Now we state three properties of the shortest path between vertices of different subpolygons. Before we state these properties in the following lemmas, we need the definition of an eave. An edge $v_i v_j$ of

$SP(v_k, v_m)$ is an *eave* if $SP(v_k, v_m)$ makes a right (or left) turn at v_i and makes a left (respectively, right) turn at v_j where v_i, v_j, v_k and v_m are distinct vertices.

Lemma 4: *If st is a visibility chord of a polygon, then the shortest path between two vertices in the same subpolygon of st has no eaves.*

Lemma 5: *If the shortest path between two arbitrary vertices in a weak visibility polygon has an eave, then every visibility chord of the polygon intersects the eave.*

Lemma 6: *In a weak visibility polygon, the shortest path between two arbitrary vertices has at most one eave.*

It can be seen that if the shortest path between any two vertices of a polygon has no eave, then the polygon need not be a weak visibility polygon (for example, a spiral polygon).

3. An algorithm for computing the maximum hidden set

The maximum hidden set of a polygon is the maximum cardinality set of vertices such that no two vertices of the set are mutually visible. The problem of finding the maximum hidden set in an arbitrary polygon is known to be NP-hard [S87]. In this section, we propose an $O(n^2)$ time algorithm for the following problem. Given a polygon P weakly visibile from a convex edge, compute the maximum hidden set of P.

Without loss of generality, we assume that P is weakly visible from the convex edge $v_n v_1$. For any two vertices v_i and v_j where $1 \leq j \leq i \leq n$, the maximum hidden set of *opposite chain* $(v_i, v_j, v_n v_1)$ is denoted as $mhs(i,j)$. The cardinality of $mhs(i,j)$ is denoted as $smhs(i,j)$. In the following lemma, we present the main idea used in the algorithm.

Lemma 7: *Assume that P is weakly visible from the convex edge $v_n v_1$. For any two vertices v_i and v_j where $1 \leq j \leq i \leq n$, $smhs(i,j) = max(smhs(i,k+1) + smhs(k-1,j), smhs(i-1,j))$ (Figure 3), where v_k is the vertex following v_i in $SP(v_i, v_j)$.*

Corollary : *If v_j is the vertex following v_i on $SP(v_i, v_j)$ then $smhs(i,j) = max(smhs(i,j+1), smhs(i-1,j))$.*

Lemma 7 suggests a simple procedure for computing the maximum hidden set of P using dynamic programming. It can be seen that $mhs(n,1)$ is the maximum hidden set of P. In the algorithm, for all values of i and j where $1 \leq j \leq i \leq n$, $smhs(i,j)$ is computed using Lemma 7 and is stored in location $A(i,j)$ of the array A. We trace the path of computation backward from the location $A(n,1)$ and reach a set of locations of the array. Each such location corresponds to a vertex of P and these vertices together form the maximum hidden set of P.

Theorem 2: *Given an n-vertex polygon P weakly visible from a convex edge, the algorithm Hidden-set finds the maximum hidden set of P in $O(n^2)$ time.*

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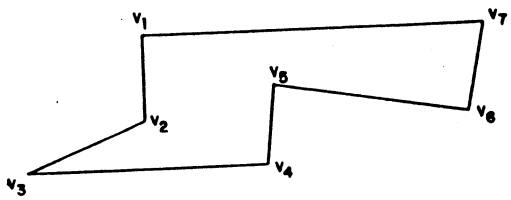

Figure 1.

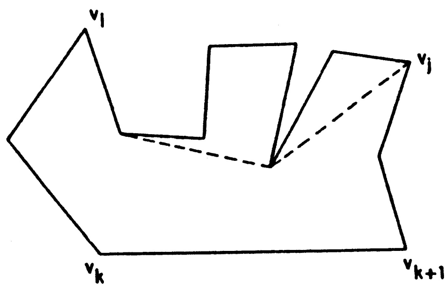

Figure 2.

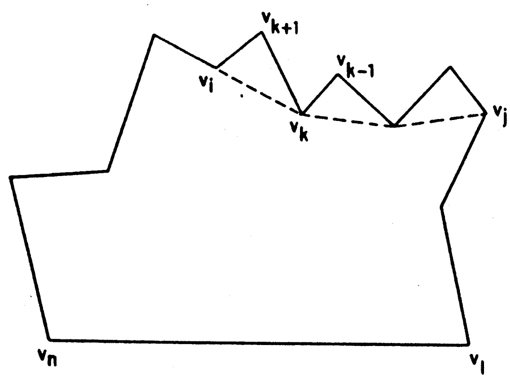

Figure 3.