WATCHMAN ROUTES UNDER LIMITED VISIBILITY

Extended Abstract

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Abstract: We discuss two versions of the watchman route problem when there is a vision range, d, for the watchman.

I. Introduction. Path planning and visibility are central problems in Computational Geometry and Robotics. Most research on visibility considers stationary vision. There has been considerable research on the Art Gallery problem and its variations [7]. In many applications we are interested on the visibility properties of a route followed by a moving guard (watchman). The watchman route problem [1,2] asks for the shortest route from a point s back to itself and with the property that each point in a given space is visible from at least one point along the route.

The commonly used definition of visibility allows for unlimited visibility along an unobstructed line of sight. This is rarely the case in practice. We say that two points are **d-visible** if they are visible and the distance between them is at most d. Many of the standard visibility algorithms can be easily modified to account for d-visibility by clipping either before or after the standard algorithm is applied (e.g., algorithms to find visibility polygons). Others reduce to variations of problems that have been considered before (e.g., the Art Gallery problem becomes the problem of covering a polygon with circles of radius d which is related to work on packing and covering with uniform shapes [9] and the disc-cover problem [6].

In the watchman route problem, a route from which the boundary of the polygon is visible is also a route from which every point in the polygon is visible. This is no longer true under d-visibility. Thus we obtain two problems. If we only want to see the boundary of the polygon we have the **d-watchman problem** which reduces to finding a shortest route that visits a set of circular sectors of radius d centered at the vertices of P. In section II, we consider the **safari route problem** which is to find a shortest route that visits a set of convex polygons that lie in the interior and are attached to the boundary of P. We present an $O(n^3)$ algorithm to find a shortest safari route by making use of the quadratic algorithm in [3] for the zoo-keeper route problem (the route visits but does not enter the polygons). In section III, we use this algorithm to obtain a polynomial approximation scheme for the d-watchman problem by modeling circles of radius d with inscribed regular k-gons. The approximation algorithms arrive at very good solutions (within 3% of optimum for $k = 6$) for all but some degenerate cases in which the length of the shortest route is very small. In section IV we discuss the watchman route problem under d-visibility when the watchman wants to see all points in P (both boundary and interior). This is equivalent to the problem of optimally sweeping a polygonal floor with a circular broom of radius d. We refer to it as the **d-sweeper problem**. It is related to the traveling salesperson problem (TSP) on simple (full) grids. The complexity of TSP is an open question for simple grids but the problem is NP-hard for general grids and for weighted grids. We present an approximation algorithm that obtains solutions to both the TSP and the d-sweeper problem that are within 33% of the optimum.

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II. The Safari Route Problem. Let P be a simple polygon and let P' be a collection of convex polygons that lie in the interior of P and are attached (i.e., they share at least one vertex) to the boundary of P. Let s be a point on the boundary of P. The **safari route problem** is to find a shortest route in the interior of the polygon from s back to s and such that the route visits each polygon in P'. We say that a route R visits a polygon P_i if the route properly intersects the polygon or shares at least one point with its boundary without entering it.

Safari routes are related to the **zoo-keeper route** problem where we want to find a shortest path that visits but does not enter each of the polygons in P'. The general zoo-keeper route problem (the polygons in P' are located arbitrarily within P) is NP-hard [3]. The same holds for the general safari route problem (by a simple reduction from Geometric Travelling Salesman [8]). An $O(n^2)$ algorithm that finds a shortest zoo-keeper route (if one exists) for the case when all the polygons in P' are attached to the perimeter of P is given in [3].

A shortest zoo-keeper route will **reflect** on some polygons in P' (i.e., have a single point contact with a polygon at which the route will turn; if the point is interior to an edge, the route behaves like a light ray hitting a mirror) and it will **wrap** around others. Wrapping is a result of the requirement that a zoo-keeper route should not enter the interior of any polygon in P'.

Let the polygons in P' be indexed according to the order implied by a clockwise scan of the boundary of P starting at $s = P_0 = P_{k+1}$. We can show that there is a shortest safari route that visits the polygons in P' on which it reflects in the order of their indices. Consider two polygons P_i, P_j in P' with $j > i$ and find their supporting segment (towards the interior of P). By looking at $j = i+1, i+2$, we can simplify the safari route problem by decomposing it into a set of safari path problems (when a supporting segment intersects the exterior of P) and/or eliminating some of the polygons in P' (when the supporting segment for P_i, P_{i+2} intersects P_{i+1}).

Suppose that P' is such that no supporting segment for P_i, P_{i+1} intersects the boundary of P and no supporting segment for P_i, P_{i+2} intersects P_{i+1}, $1 \leq i \leq k$. Consider the safari route problem for such P, P' (the safari path problems that result from decomposition can be solved the same way). Construct a shortest zoo-keeper route R for P, P'. We have:

Lemma 1: If R reflects on all polygons in P', then R is a shortest safari route for P, P'.

If R does not reflect on all P_i in P' we form the sets P_r, P_w of polygons in P' that R reflects on and wraps around respectively and we index these sets in a clockwise order. We partition the set P_w into two subsets, the **even wrap set** and the **odd wrap set** (depending on whether the previous reflection was at an even/odd indexed polygon in P_r). We construct two shortest zoo-keeper routes, route R_{even} that visits the set $P_r \cup P_{w-even}$ (disregarding the set P_{w-odd}) and route R_{odd} that visits the set $P_r \cup P_{w-odd}$ (disregarding P_{w-even}). Each of these routes will go though some polygons in the excluded set (since we disregard some polygons when we construct each of R_{w-even}, R_{w-odd}). If a route intersects a polygon, we measure the **dislocation** of the route with respect to the polygon as the distance between the segment of the route that goes through the polygon and the supporting line parallel to it.

Lemma 2: Let $P_{\max}$ be the polygon where the maximum dislocation occurs (with respect to R_{w-even} or R_{w-odd}). Then, the shortest safari route will always go through $P_{\max}$, and we can remove it from the set P'.

Each time the above process is repeated, we can eliminate one polygon in P' from further consideration (i.e., the final route will go through it). Eventually, the shortest zoo-keeper route for the remaining set will reflect on all polygons and we have a shortest safari route.

Theorem 1: A shortest safari route can be constructed in $O(mn^2)$, where m is the size of P' and n is the total number of vertices in P, P'.

III. The d-watchman problem. In this section we consider the watchman route problem in a simple polygon when the visibility range of the watchman is d and we are only interested in viewing the boundary of the polygon. We refer to this problem as the **d-watchman problem.**

Consider a set of n circular sectors of radius d centered at each vertex of P. The circular sectors lie in the interior of P and are bounded by the two edges adjacent to each vertex. Note that each circular sector can be viewed as a convex polygon.

Lemma 3: Any shortest safari route that visits the set of circular sectors in P is also a shortest d-watchman route for P.

To find an exact shortest d-watchman route, we can try to solve the problem of finding the shortest route that visits the set of circular sectors. Many of the circular sectors can be eliminated using arguments similar to those used in the safari route problem (e.g., all the circular sectors associated with reflex vertices in P can be disregarded). However, we eventually need to solve the problem of finding the shortest path that reflects on a set of circular sectors. Using local optimality criteria for the path one can derive a function describing the path length and solve for the best contact point by differentiating to find the minimum. In the general case where we have $O(n)$ circles, the length function and the local optimality criteria reduce to a high order equation which seems to require numerical solutions.

We consider approximate solutions by modeling the circular sectors with inscribed regular polygons of k sides (for the whole circle) and solving the resulting safari route problem. The analysis of the quality of the solutions falls apart in cases where the circular sectors are highly overlapping so that the shortest d-watchman route can be arbitrarily small. Such cases are rare (we refer to them as degenerate cases) and can be treated separately using numerical methods.

Theorem 2: In non-degenerate cases, if we inscribe regular k-gons in each circle, the shortest safari route can be computed in $O(kn^3)$ and it will be within $(1+[(1-\cos\theta)/\tan\theta]^2)^{1/2}$ of the shortest d-watchman route ($\theta = 360/k$). (For $k = 6, 20$ the approximate route is at most 3%, 1.2% longer than the optimum respectively).

IV. The d-sweeper problem. In this section we consider the watchman route problem for simple polygons when there is a visibility range of d and we are interested in viewing all of P (i.e., both its interior and its boundary). We refer to the resulting problem as the **d-sweeper problem** because it is equivalent to finding the "least work" way to sweep a polygonal floor with a circular broom of radius d.

The problem of sweeping the floor with the least amount of work has been considered in [4]. They discuss general strategies but do not analyze the quality of their heuristics. The problem is also related to the traveling salesman problem in grids. We can superimpose a grid of unit size $2d$ on the polygon, clip portions of the grid on the exterior of the polygon and ask for the shortest traveling salesman route that visits all vertices of the grid. Depending on how fine the grid is (i.e., the relation between d and the dimensions of the polygon), the grid solution will be very close to optimum (the finer the grid, the better the solution). We will assume that the unit size of the grid is much smaller than the average length of polygon edges.

It is known that the Hamiltonian Path problem on grids (and hence the TSP problem) is NP-hard for grids that may have missing sections [5]. It is also known that the TSP is NP-hard for weighted simple grids but its complexity is open for simple grids with unit weights.

Our approach to solving the d-sweeper problem is to superimpose a grid on the polygon, remove the portions of the grid that lie on the exterior of the polygon and find a TSP route on the resulting simple grid. A solution to TSP should try to minimize the number of times the route visits grid vertices that have already been visited. We construct a TSP route by partitioning the original simple grid into rectangular subgrids, finding TSP routes for each subgrid and then connecting the partial routes into one TSP route that visits all the vertices in the original grid. Because the original grid is simple, we can partition the grid so that the partition will have a tree-like structure. This structure allows us to designate boundaries where the outer rectangular grids will attach to inner ones. So that the attachments of the various partial routes can be easily performed (without incurring any additional cost from visiting vertices more than once), we restrict the route for each subgrid to have the property that it contain a path that follows the boundary except possibly where the subgrid attaches to an interior subgrid. We call such a TSP route a **wrap route.**

Lemma 4: For any simple rectangular $m x n$ grid, $m,n > 1$, there is a wrap route with length at most 33% more than mn.

Note that the 33% comparison is made with the length of a (possibly non-existent) Hamiltonian circuit for the rectangular grid. Since this length is a lower bound on the length of any TSP route, we conclude that the wrap route is within 33% of the optimum. Note that Lemma 4 does not allow $m,n = 1$. If we have a strip of width one that attaches to the rest of the grid at only one vertex, any TSP route must visit the vertices along the strip twice (no Hamiltonian circuit exists). Note that the wrap paths allow the attachments between adjacent rectangular subgrids to be made at no additional cost.

Theorem 3: Combining wrap routes yields a TSP route that is within 33% of the optimum.

If d is small with respect to the size of the polygon, we can make local adjustments to the TSP route for a superimposed grid and obtain an approximate solution to the d-sweeper problem that is within 33% of the optimum. The 33% bound is not tight and we expect to improve on it.

References.

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