Decomposing a Star Graph into Disjoint Cycles

K. Qiu H. Meijer S. Akl

Department of Computing and Information Science
Queen's University
Kingston, Ontario, Canada

1. Introduction

The star graph has been proposed in 1986 [1] as an attractive alternative to the n-cube. As a new interconnection topology, it possesses rich structure and symmetry properties as well as many desirable fault tolerant characteristics [1,2,3] and it compares favorably with the n-cube in many aspects. In this paper, we show that an n-star can be decomposed into $(n-2)!$ disjoint cycles of length $(n-1)n$. These cycles may be used as basic unit in designing algorithms on star graphs.

This paper is organized as follows. In Section 2, we give the definition as well as some basic properties of an n-star. Section 3 describes the decomposition of an n-star into disjoint cycles.

2. The Star Graph

Given a set of generators for a finite group G, the *Cayley graph* with respect to G is defined as follows. The vertices of the graph correspond to the elements of the group G, and there is an edge (a, b) for $a, b \in G$ if and only if there is a generator g such that $ag=b$. We require that the set of generators be closed under inverse so that the resulting graph can be viewed as being undirected [1].

Let G be a permutation group, we represent a permutation by $a_1a_2 \cdots a_n$, where $a_i \in \{1, 2, ..., n\}$ and $a_i \neq a_j$ if $i \neq j$. A star graph on n symbols or an n-star is a Cayley graph of $n!$ nodes with generators $\{i23...(i-1)1(i+1)...n \mid 2 \leq i \leq n\}$. The vertex set V is the set of all $n!$ permutations on n symbols. We denote an n-star by S_n (Fig. 1). From the definition of an n-star we know that S_n is a regular graph of degree $n-1$. Each node (permutation) in S_n is connected to $n-1$ nodes (permutations) which can be obtained by interchanging the first symbol of the node with the i^{th} symbol, $i=2, 3, ..., n$. We call these $n-1$ connections the dimensions. Thus, any node in S_n is connected to $n-1$ nodes along dimensions i, $i=2, 3, ..., n$. For any node $A=a_1a_2 \cdots a_n$ in S_n, define functions f and l as follows:

$$f(a_1a_2 \cdots a_n)=a_1,$$
$$l(a_1a_2 \cdots a_n)=a_n.$$

The vertices of S_n can be partitioned into n groups, $S_{n-1}(i)$, $1 \leq i \leq n$. Each $S_{n-1}(i)$ is defined as the subgraph of S_n induced by all vertices A with $l(A)=i$. It can be seen that $S_{n-1}(i)$ is a $(n-1)$-star.

This work is supported by the Natural Sciences and Engineering Research Council of Canada (grants A-0282 and A-3336), and by the School of Graduate Studies and Research, Queen's University, Canada.

For example, S_4 in Fig. 1 contains four 3-stars $S_3(1)$, $S_3(2)$, $S_3(3)$, and $S_3(4)$.

3. Decomposing a Star Graph into Disjoint Cycles

In this section, we give a partition which decomposes an n-star into disjoint cycles.

We begin with an example. Assume that $d_i=i$, $i=2, 3, ..., n$. Let the starting node be $a_1a_2 \cdots a_n$. If we visit the nodes along the dimensions 2, 3, ..., and n, repeatedly, starting from starting node $A_1=a_1a_2 \cdots a_n$, we get a cycle $A_1, A_2, ... A_{(n-1)n}$ as follows:

A_1: $a_1a_2a_3....a_n$ $A_{(n-1)+1}$: $a_na_1a_2....a_{n-1}$ ...... $A_{(n-1)(n-1)+1}$: $a_2a_3a_4....a_1$

A_2: $a_2a_1a_3....a_n$ $A_{(n-1)+2}$: $a_1a_na_2....a_{n-1}$ ...... $A_{(n-1)(n-1)+2}$: $a_3a_2a_4....a_1$

A_3: $a_3a_1a_2....a_n$ $A_{(n-1)+3}$: $a_2a_na_1....a_{n-1}$ ...... $A_{(n-1)(n-1)+3}$: $a_4a_2a_3....a_1$

... ...

A_{n-1}: $a_{n-1}a_1a_2..a_n$ $A_{2(n-1)}$: $a_{n-2}a_na_1..a_{n-1}$ ..... $A_{(n-1)n}$: $a_na_2a_3..a_{n-1}a_1$

All the nodes are easily seen to be distinct, thus the length of the cycle is $(n-1)n$.

Lemma 1. Let $d_2d_3 \cdots d_n$ be a permutation of the symbols 2, 3, ..., n. In S_n, if we start from an arbitrary node and visit nodes along dimensions d_2, d_3, ..., d_n, d_2, d_3, ... etc. repeatedly, then we get a cycle C of length $(n-1)n$ such that $C \cap S_{n-1}(i)$ is a path containing $n-1$ vertices for $i=1, 2, ..., n$.

Proof. If $d_n=n$, it is easy to see that for any permutation $d_2d_3 \cdots d_n$, once the cycle goes into a $(n-1)$-star $S_{n-1}(i)$ in S_n for some i, it visits $n-1$ nodes in this $S_{n-1}(i)$ before it goes out along dimension $d_n=n$. So $C \cap S_{n-1}(i)$ is a path of length $n-1$. In case $d_i=n$, $2 \leq i < n$, we can cyclically shift permutation $d_2d_3 \cdots d_{i-1}nd_{i+1}...d_n$ to $d_{i+1} \cdots d_nd_2d_3...d_{i-1}n$ and pick a node A' as the new starting point, which is obtained by visiting nodes along dimensions $d_2d_3...d_{i-1}n$ starting from $A=a_1a_2 \cdots a_n$. Using the new starting point and new dimensions $d_{i+1} \cdots d_nd_2d_3...d_{i-1}n$ we get a cycle which is similar to the above one. □

We will now describe a set of starting points of the $(n-2)!$ cycles. For each starting point we also define the set of dimensions that are needed to generate the cycle. We will then prove that these $(n-2)!$ cycles of length $(n-1)n$ are disjoint.

Let a starting point be $1*n$, where * is a permutation of symbols in $\{2, 3, ..., (n-1)\}$. With each starting point of the form $1*n$, we will associate a unique permutation $d_2d_3 \cdots d_{n-1}n$. By Lemma 1, node $1*n$ together with $d_2d_3...d_{n-1}n$ generates a cycle of length $(n-1)n$ in S_n. We call D_T = $d_2d_3 \cdots d_{n-1}n$ the permutation associated with the starting point $T=1*n$. We denote the cycle generated by the starting point T and its associated permutation D_T by $C(T, D_T)$. For any starting point of the form $T=1*n$, the permutation $D_T=d_2d_3...d_{n-1}n$ is defined as follows. Let P_T be a function such that for $T=1t_2t_3 \cdots t_{n-1}n$,

$$P_T(i)=j \text{ if and only if } t_j=i,$$

then

$$d_2d_3 \cdots d_{n-1} = P_T(2)P_T(3)...P_T(n-1).$$

From this definition, we can see that the cycle $C(T, D_T) = V_1, V_2, ..., V_{(n-1)n}$ with $T=V_1=1{*}n$ has the form as shown in Fig. 2, i.e.,

$$f(V_i) = \begin{cases} n & \text{if } n \mid i \\ i \bmod n & \text{otherwise} \end{cases} \tag{1}$$

and

$$l(V_i) = n - \lfloor \frac{i-1}{n-1} \rfloor. \tag{2}$$

Lemma 2. For two starting points U_1 and V_1 of the form $1{*}n$ in S_n with $U_1 \neq V_1$, we have

$$C(U_1, D_{U_1}) \cap C(V_1, D_{V_1}) = \emptyset. \tag{3}$$

Proof: Assume $U_1 \neq V_1$. Let the cycle $C(U_1, D_{U_1})$ be $U_1, U_2, ..., U_{(n-1)n}$ and the cycle $C(V_1, D_{V_1})$ be $V_1, V_2, ..., V_{(n-1)n}$. Suppose $C(U_1, D_{U_1}) \cap C(V_1, D_{V_1}) \neq \emptyset$, so $U_i = V_j$ for some i and j. We have

$$l(U_i) = l(V_j) = n - \lfloor \frac{i-1}{n-1} \rfloor = n - \lfloor \frac{j-1}{n-1} \rfloor, \tag{4}$$

and

$$f(U_i) \equiv f(V_j) \equiv i \equiv j \bmod n. \tag{5}$$

Therefore $i = j$. Let

$$U_i = V_i = a_1 a_2 \cdots a_{n-1} a_n$$

for some $i > 1$. If $a_1 > 1$ we have

$$U_{i-1} = (a_1 - 1) a_2 .. a_1 .... a_n$$
$$V_{i-1} = (a_1 - 1) a_2 .... a_1 .. a_n$$

otherwise

$$U_{i-1} = n a_2 .. a_1 .... a_n$$
$$V_{i-1} = n a_2 .... a_1 .. a_n.$$

So $U_{i-1} = V_{i-1}$. Repeating this argument, we derive that $U_1 = V_1$, a contradiction. Hence the Lemma. □

We are now ready to state our main result of the paper.

Theorem 1. An n-star S_n can be decomposed into $(n-2)!$ disjoint cycles of length $(n-1)n$.

Proof: There are $(n-2)!$ starting points of the form $1{*}n$ in an n-star S_n, where $*$ is a permutation of $\{2, 3, ..., n-1\}$. Each of them uniquely determines its associated permutation $d_2 d_3 \cdots d_{n-1} n$. By Lemmas 1 and 2, we have $(n-2)!$ disjoint cycles of length $(n-1)n$. □

Fig. 3 shows two cycles in S_4.

References

[1] S.B. Akers and B. Krishnamurthy, "A Group Theoretic Model for Symmetric Interconnection Networks," *Proc. ICPP,* 1986, pp.216-223. Also in *IEEE Trans. on Compu.* Vol. 38, No. 4, 1989.

[2] S.B. Akers, D. Harel, and B. Krishnamurthy, "The Star Graph: An Attractive Alternative to the *n*-cube," *Proc. ICPP,* 1987, pp.393-400.

[3] S.B. Akers and B. Krishnamurthy, "The Fault Tolerance of Star Graphs," 2[nd] *International Conf. on Supercomputing,"* San Francisco, CA, May 1987.

[4] S.G. Akl, *The Design and Analysis of Parallel Algorithms,* Prentice Hall, Englewood Cliffs, New Jersey, 1989.

[5] K. Qiu, H. Meijer, and S.G. Akl, "Decomposing a Star Graph into Disjoint Cycles", Technical Report No. 90-278, Department of Computing and Information Science, Queen's University, Kingston, Canada, May 1990.

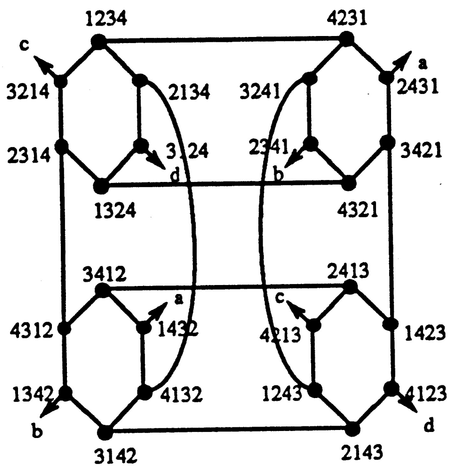

Figure 1. A 4-Star.

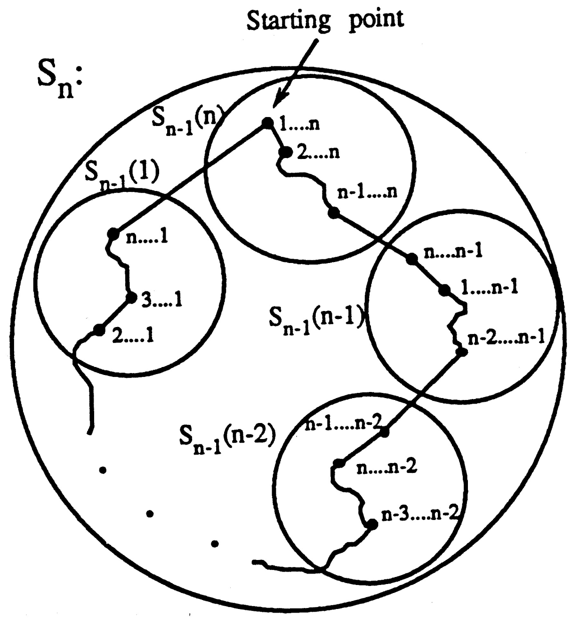

Figure 2. A Cycle in S_n.

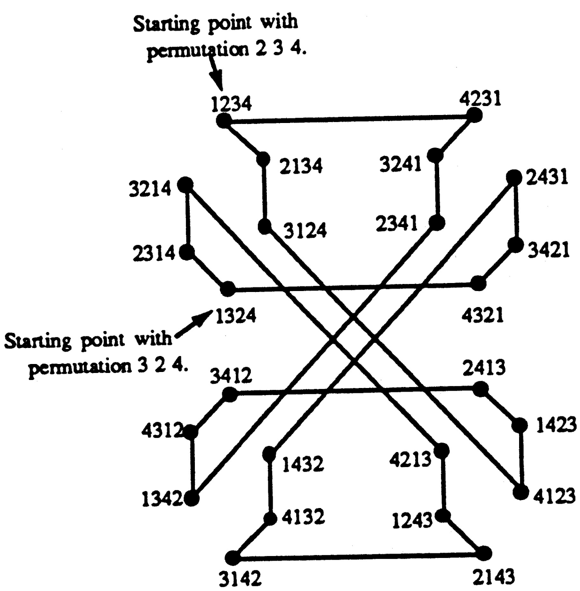

Figure 3. Two Cycles in S_4.