Topology-Oriented Approach to Robustness and Its Applications to Several Voronoi-Diagram Algorithms

Kokichi Sugihara*, Y. Ooishi, and T. Imai
Department of Mathematical Engineering and Information Physics
Faculty of Engineering, University of Tokyo
7–3–1 Hongo, Bunkyo-ku, Tokyo 113, Japan

The paper presents a new scheme for constructing geometric algorithms robust against numerical errors, and apply the scheme to several algorithms for constructing Voronoi diagrams.

Conventionally geometric algorithms are designed in the assumption that numerical values are computed precisely. In real computers, on the other hand, numbers are represented and computed in finite precision, so that we cannot always decide topological relations correctly. For example, it is sometimes difficult to judge whether a point is to the right of, to the left of, or exactly on a line, particularly when the point is very close to the line. Misjudgements on the topological structure often create inconsistency, and cause a theoretically correct algorithm to fail. Hence, there is a great gap between theoretically correct algorithms and practically valid computer programs.

In order to fill this gap, we start with the assumption that numerical errors take place in the course of computation and also that the amount of errors cannot be bounded. In such an imprecise world, it is impossible to construct a correct

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* The first author is tentatively at Department of Computer Science, Purdue Univeristy, West Lafayette, IN 47907, U. S. A.

algorithm. Hence, we change our goal; we try to find an approximation algorithm that is at least robust against numerical errors.

Our key idea for constructing a robust algorithm is "combinatorial abstraction", if we name it. That is, we first abstract a purely combinatorial structure from a geometric procedure, and describe the fundamental part of the algorithm in terms of combinatorial computations alone. This kind of a description of the algorithm is free from numerical errors because numerical computations are not employed. However, there is usually ambiguity in the choice of branches of the procedure, and in order to resolve the ambiguity, and for this purpose only, we employ numerical computations. An algorithm designed in this way has the following properties:

(i) the algorithm is robust against numerical errors in the sense that, no matter how large numerical errors take place, the algorithm never comes across inconsistency and always gives some output,

(ii) the algorithm is correct in the sense that the output converges to the correct answer as the precision in computation becomes higher, and

(iii) the algorithm is simple in the sense that it does not have any special branch for degeneracy.

In the First Canadian Conference on Computational Geometry, one of the authors took this design principle in order to construct an incremental-type algorithm for the Voronoi diagram in the plane [1]. In the present paper, we apply the same principle to two other algorithms: a divide-and-conquer algorithm for constructing Voronoi diagrams for points, and an incremental-type algorithm for constructing Voronoi diagrams for line segments.

It was known that the divide-and-conquer algorithm runs in $O(n \log n)$ time on the average, as well as in the worst case, for n generators, but recently it was shown that the algorithm runs in $O(n)$ time on the average if the region containing the generators is divided into square subregions [2]. This average time complexity is the same as that of the incremental-type algorithm, so that it is worth trying to make the divide-and-conquer algorithm robust and to compare it with the incremental-type robust algorithm.

With this motivation, we applied our principle to the divide-and-conquer algorithm [3]. We actually considered the construction of the Delaunay diagram, dual of the Voronoi diagram. In our new algorithm, we present a set of topological conditions that should be satisfied by the merge process, and place higher priority on these conditions than on numerical values. One of the conditions, for instance, is that "parallel Delaunay edges should not exist between any pair of generators". The resultant algorithm is robust; for example, the algorithm could construct in single-precision arithmetic a diagram approximating the Voronoi diagram for 2,000 generators placed on a common circle, for which conventional algorithms usually fail in processing because it is a highly degenerate set of generators. Moreover, our algorithm attains an almost linear average-case time complexity.

In the second application, we construct an incremental algorithm for the Voronoi diagrams for line segments. In this algorithm, we first decompose the input line segments into terminal points and open line segments, next employ our robust algorithm for constructing the Voronoi diagram for the terminal points, and finally change the diagram by adding open line segments one by one. In the addition of a new line segment, we place the highest priority on the topological condition that "the substructure of the diagram to be replaced by the Voronoi region of the new line segment is a tree", and thus attain robustness.

In our approach we concentrate our attention on achieving robustness without increasing time complexity. However, there is another important point, that is, stableness. Sometimes an algorithm is said to be stable if the output of the algorithm is the correct answer to some perturbation of the input, but it seems very difficult to make our algorithms stable in this sense because it is a hard problem to judge whether a given diagram is topologically isomorphic to a Voronoi diagram for some set of generators. Hence, we are "practically" aiming at stableness by choosing the ways of numerical computations carefully. "Theoretical" consideration of stableness of our algorithms is one of problems in future.

References

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