A Unified Linear-Space Approach to Geometric Minimax Problems

(Extended Abstract)

Tetsuo Asano[1], Michael E. Houle[2], Hiroshi Imai[3] and Keiko Imai[4]

[1]Osaka Electro-Communication University, Neyagawa 572, Japan

[2]Department of Computer Science and Communication Engineering,
Kyushu University, Fukuoka 812, Japan,

[3]Department of Information Science, University of Tokyo, Tokyo 113, Japan,

[4]Department of Mathematics, Tsuda College, Tokyo 187, Japan.

1 Introduction

In 1982, Comer and O'Donnell formulated and solved a geometric problem applicable to the choice of good hashing functions. Given a set S of points in the plane, they wished to find a line l such that the *span* of the orthogonal projection of S on l (the maximum distance between projected points) divided by the *resolution* (the minimum distance between distinct projected points) is minimized [CO82]. In their paper, Comer and O'Donnell presented an $O(n^2 \log n)$ time and $O(n^2)$ space algorithm to solve this geometric problem.

Recently, Huttenlocher and Kedem considered the problem of finding the minimum Hausdorff distance between two sets of points on the real line, over all possible translations of the sets [HK90]. Given two (static) sets of points A and B, the Hausdorff distance between them is defined as

$$\max\{\max_{a\in A}\min_{b\in B}\delta(a,b), \max_{b\in B}\min_{a\in A}\delta(a,b)\},$$

where $\delta(a,b)$ is simply the Euclidean distance between a and b. This minimum Hausdorff distance, together with the amount by which the sets must be translated to achieve it, provide information which may be useful in solving certain clustering problems. Their solution requires $O(n_a n_b \log n_a n_b)$ time and $O(n_a n_b)$ space, where n_a and n_b are the cardinalities of A and B, respectively.

On the surface, these problems may seem quite unrelated. However, one characteristic common to both is that the minima of distance functions are being maximized (or maxima are being minimized). Such problems are often referred to as "minimax" problems in the literature. In this paper, we shall show how these and other minimax problems may be transformed into one of a family of minimax problems on line arrangements. By providing a unified solution to this family of problems, a wide range of minimax problems may be solved. These will be seen to have applications in such areas as hashing, clustering, and VLSI design.

The unified approach involves sweeping line arrangements while simultaneously constructing upper and lower envelopes of a quadratic number of "V-shaped" functions. Although these envelopes ordinarily require storage space quadratic in the input size, only a portion of each envelope will be kept on hand at any given time. In this way, the space complexities of the problems (including those of [CO82] and [HK90] mentioned earlier) may be made linear in the input size, without any increase in asymptotic worst-case time complexity.

In this extended abstract, we will not be able to give the details of our method. Instead, in the next section, we state the general line arrangement problem, and give a sketch of its solution. In Section 3, we show how the problems stated earlier may be expressed as a problem on line arrangements.

2 The Line Arrangement Problem

Let $L = \{l_1, l_2, \ldots, l_n\}$ be a set of n distinct non-vertical lines in the plane, where each l_i is parameterized as $l_i(t) = a_i t + b_i$, for all $0 \le i \le n$. Let us also consider a collection of "colours" $C = \{1, 2, \ldots, c\}$ such that each line l of L is assigned some colour $\chi(l) \in C$. This assignment of colours to lines partitions L into the disjoint subsets $L_j = \{l \in L \,|\, \chi(l) = j\}$, for all $0 \le j \le c$. Let n_j be the cardinality of L_j.

The main focus of our paper will be the minimization (or maximization) over t of functions of the form

$$F(t) = f(t, \phi_1(t), \phi_2(t), \ldots, \phi_m(t)),$$

where each ϕ_k is the upper or lower envelope of a collection of at most n^2 "V-shaped" functions of t, expressed in terms of the values of the lines of L evaluated at t. The function F must be one whose local maxima and minima may be computed analytically (the function need not be bounded).

One of the ways in which such problems may be solved is to represent each of the ϕ_k as a chain of line segments, and then merge these chains while simultaneously computing the local minima (or maxima) of F. In the full paper, we shall see that the construction of these functions may be performed by way of a line sweep of the line arrangement induced by L. Since some of these functions may require $\Omega(n^2)$ storage, we shall also describe a general method by which these functions may be constructed for intervals of values of t, such that the size of each function over the interval is linear in n. By processing a linear number of such intervals, we may evaluate F at all local minima (or maxima) in $O(I(n)\log I(n) + n\log n)$ time and $O(n)$ space, where $I(n)$ is the number of intersections between lines of the arrangement.

3 Applications

There are of course a great many functions ϕ_k from which to choose; the most interesting of these involve the vertical spacings of the lines of L. As the value of t varies between $-\infty$ and ∞, the vertical ordering and spacing of the lines also varies. Let $L^t = (l_1^t, l_2^t, \ldots, l_n^t)$ be the sequence of lines of L sorted in increasing order of their values at t. ~~for each colour j~~. Obviously, for every t, $i < j$ implies that $l_i^t(t) \leq l_j^t(t)$.

Consider now the following functions of t:

$$\begin{aligned}
\phi_a(t) &= \min_{1\leq i<n} \{l_{i+1}^t(t) - l_i^t(t)\} \\
\phi_b(t) &= l_n^t(t) - l_1^t(t) \\
\phi_c(t) &= \max_{1\leq j\leq n} \min_{\substack{1\leq i\leq n \\ \chi(l_j^t)\neq\chi(l_i^t)}} \{|l_j^t(t) - l_i^t(t)|\} \\
\phi_d(t) &= \min_{\substack{1\leq i,j\leq n \\ \chi(l_j^t)\neq\chi(l_i^t)}} \{|l_j^t(t) - l_i^t(t)|\}
\end{aligned}$$

It may easily be verified that each is the upper or lower envelope of a subset of the set of functions $V = \{v_{i,j} | v_{i,j}(t) = |l_i(t) - l_j(t)|\}$, for $1 \leq i,j \leq n$. The first of these functions may be interpreted as the minimum vertical distance between two consecutive lines of L evaluated at t; the second may be interpreted as the maximum vertical distance between any two lines. Function ϕ_c gives the maximum vertical distance between a line and the closest line of a differing colour. Function ϕ_d determines the minimum vertical distance between two lines of differing colours.

The two problems stated in the introduction may be stated in terms of these functions of t. For a set S consisting of the points (x_i, y_i) for $1 \leq i \leq n$, Comer and O'Donnell's problem may be transformed into the following:

$$\text{minimize}_t \qquad \frac{\phi_b}{\phi_a}$$

where $L = \{l_1, l_2, \ldots, l_n\}$ such that $l_i(t) = x_i t + y_i$. This problem may thus be solved in $O(n^2 \log n)$ time and $O(n)$ space.

The problem of finding the minimum Hausdorff distance between two sets $A = \{a_1, a_2, \ldots, a_{n_a}\}$ and $B = \{b_1, b_2, \ldots, b_{n_a}\}$ and on the line, under translation, may be expressed as

$$\text{maximize}_t \qquad \phi_C$$

where $L = L_1 \cup L_2$, $L_1 = \{l_{1,1}, l_{1,2}, \ldots, l_{1,n_a}\}$ such that $l_{1,i}(t) = a_i$, and $L_2 = \{l_{2,1}, l_{2,2}, \ldots, l_{2,n_b}\}$ such that $l_{2,i}(t) = t + b_i$. Since there are only $n_a n_b$ mutually-intersecting pairs of lines of L, a solution may be obtained in $O(n_a n_b \log n_a n_b)$ time and $O(n_a + n_b)$ space.

Another problem which may be cast into the same form shall be seen in the full paper to have possible applications in VLSI design. Simply stated, given two sets of planar points $A = \{(x_{a,1}, y_{a,1}), (x_{a,2}, y_{a,2}), \ldots, (x_{a,n_a}, y_{a,n_a})\}$ and $B = \{(x_{b,1}, y_{b,1}), (x_{b,2}, y_{b,2}), \ldots, (x_{b,n_b}, y_{b,n_b})\}$, find a line l such that the minimum distance between any point of the orthogonal projections of A on l, and any point of the orthogonal projection of B on l, is maximized. This problem may be viewed as that of connecting one set of terminals (the points of A) by parallel wires leading in one direction, and connecting the remaining terminals (the points of B) by parallel wires leading in the opposite direction, such that the minimum distance between wires leading to terminals of different sets is maximized.

In the arrangement setting, this routing problem becomes

$$\text{maximize}_t \qquad \frac{\phi_d}{\sqrt{1+t^2}}$$

where $L = L_1 \cup L_2$, $L_1 = \{l_{1,1}, l_{1,2}, \ldots, l_{1,n_a}\}$ such that $l_{1,i}(t) = x_{a,i} t + y_{a,i}$, and $L_2 = \{l_{2,1}, l_{2,2}, \ldots, l_{2,n_b}\}$ such that $l_{2,i}(t) = x_{b,i} t + y_{b,i}$. If $n = n_a + n_b$, we may obtain a solution in $O(n^2 \log n)$ time and $O(n)$ space.

In addition to these problems, several other problems related to hashing and clustering may also be solved under the same framework.

References

[CO82] Comer, D. and O'Donnell, M. J. Geometric problems with applications to hashing, *SIAM J. Comput.* 11(2) (1982), pp. 217-226.

[HK90] Huttenlocher, D. P. and Kedem, K. Computing the minimum Hausdorff distance for point sets under translation, *Proc. 6th Ann. ACM Symp. Comp. Geom.* (1990), *to appear.*