Computing External-Farthest Neighbors for a Simple Polygon

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Abstract

Let $\mathcal{P}$ be (the boundary of) a simple n-gon. For a pair of points $p, q \in \mathcal{P}$ consider *(external) paths* connecting them, which are the paths that avoid the interior of $\mathcal{P}$. The *(external) distance* $d(p,q)$ between two points p and q is the length of a shortest external path connecting p to q. For a given point $p \in \mathcal{P}$, an *external farthest neighbor* of p is a point $q \in \mathcal{P}$ such that $d(p,q) = \sup\{d(p,p') \mid p' \in \mathcal{P}\}$, and the *external diameter* $\mathcal{D}(\mathcal{P})$ of $\mathcal{P}$ is $\sup\{d(p,q) \mid p,q \in \mathcal{P}\}$. Samuel and Toussaint gave an $O(n^2)$ algorithm for computing $\mathcal{D}(\mathcal{P})$. We present an $O(n \log n)$ algorithm for computing an external farthest neighbor for every vertex of $\mathcal{P}$. The external diameter of $\mathcal{P}$ can also be computed in the same time bound.

For distinct points $p, q \in \mathcal{P}$, let $\mathcal{P}[p,q]$ be the section of $\mathcal{P}$ clockwise from p to q inclusive. We define a *pocket* of $\mathcal{P}$ to be a closed polygonal region bounded by an edge of the convex hull of $\mathcal{P}$ and the portion of $\mathcal{P}$ between the endpoints of this edge.

A farthest neighbor of p within the same pocket can be computed by an algorithm of Suri. Therefore, we only consider the farthest neighbors $\phi'(p)$ of a vertex $p \in \mathcal{P}$ among the points of $\mathcal{P}$ not lying in the same pocket as p. For distinct points $p, q \in \mathcal{P}$ lying in different pockets, let us define the *left* (resp. *right*) shortest path as the shortest path from p to q going around $\mathcal{P}$ in clockwise (counterclockwise) direction. Let $d_L(p,q)$ (resp. $d_R(p,q)$) denote the length of the left (resp. right) shortest path. The shorter of the above two paths is an external shortest path from p to q.

We show that for any point $p \in \mathcal{P}$ there exists a *bifurcation point* $\beta(p) \in \mathcal{P}$ such that $d_L(p,y) \leq d_R(p,y)$, for $y \in \mathcal{P}[p, \beta(p)]$ and $d_L(p,y) > d_R(p,y)$ otherwise. Therefore, if p and r are two adjacent vertices of $\mathcal{P}$, then p, r, $\beta(p)$, $\beta(r)$ occur in this cyclic order on $\mathcal{P}$ (possibly $\beta(p) = \beta(r)$). Hence $\beta(p)$, for all vertices of $\mathcal{P}$, can be determined in $O(n)$ time by first locating the bifurcation point of one vertex and then scanning $\mathcal{P}$ once more in clockwise direction to compute bifurcation points of the remaining vertices of $\mathcal{P}$.

Let $\delta_L(p)$ (resp. $\delta_R(p)$) represent a vertex of $\mathcal{P}$ farthest from p among the vertices lying in $\mathcal{P}[p, \beta(p)]$ (resp. $\mathcal{P}[\beta(p), p]$) but not in the same pocket as p, then for any point $p \in \mathcal{P}$, at least one of $\delta_L(p)$, $\beta(p)$, $\delta_R(p)$ is in $\phi'(p)$. If p and r are two adjacent vertices of $\mathcal{P}$, then p, r, $\delta_L(p)$, $\delta_R(r)$ occur in this cyclic order on $\mathcal{P}$ (possibly $\delta_L(p) = \delta_L(r)$). Once we have computed bifurcation points for all vertices of $\mathcal{P}$, we can make use of this ordering to compute δ_L and δ_R in $O(n)$ time.

As for computing the external diameter of $\mathcal{P}$, the observation that there exists a vertex p of $\mathcal{P}$ whose distance to its external farthest neighbor(s) is $\mathcal{D}(\mathcal{P})$, combined with the above discussion immediately yields an $O(n \log n)$ algorithm.