General Metrics and Angle Restricted Voronoi Diagrams

Young C. Wee *, *Seth Chaiken and Dan E. Willard* †

Department of Computer Science State University of New York at Albany Albany, New York 12222

Abstract

This paper contains results about generalizations of metrics in the plane. First, we find that the MST $\subseteq$ RNG $\subseteq$ DT inclusion, already known for L_p metrics, holds for much more general metrics. Examples include the non translation invariant geodesic metrics defined from geodesics through weighted regions or around obstacles. Second, we return to translation invariance and present results based on **angle restricted** versions of ordinary metrics and related notions. Let $VI_\alpha(p)$ denote the closed region between the two rays in directions r_1,r_2 from p forming an acute angle $\alpha = [r_1,r_2]$. For any convex *base* metric d, d_α is defined by $d_\alpha(x,y) = d(x,y)$ if $y \in VI_\alpha(x)$ and $d_\alpha(x,y) = \infty$ otherwise.

MST $\subseteq$ DT for the Euclidean metric was shown by [Shamos and Hoey 75]. The RNG is defined so for $x,y \in S$, $(x,y) \in$ RNG iff for all $z \in S-\{x,y\}$, $d(x,y) \leq d(x,z)$ or $d(x,y) \leq d(y,z)$. [Jaromczyk and Kowaluk 87] showed MST $\subseteq$ RNG $\subseteq$ DT for all L_p metrics. The proofs [Toussaint 80, Jaromczyk and Kowaluk 87] that MST $\subseteq$ RNG apply to any symmetric weight function d. We find that the RNG $\subseteq$ DT inclusion also holds for any symmetric convex metrics that have **geodesics** in the following sense: For any x,y in the domain, there exists a continuous curve $\gamma(x,y)$ between x and y such that for all $z \in \gamma$, $d(x,y) = d(x,z) + d(z,y)$.

We present an $O(N\log N)$ divide and conquer algorithm for computing the Voronoi diagram of a finite set of N points with the distance function d_α. This algorithm implies, for any convex distance function, we can compute in $O(N\log N)$ time the *all pairs geographic nearest neighbors* of a set of N sites in the plane. In [Yao 82], an $O(N^{2-1/8}\log^{2-1/8}N)$ time algorithm for this problem is given for L_1, L_2 and L_∞ metrics. For L_1 and L_∞ metrics, an $O(N\log N)$ time algorithm is given by [Guibas and Stolfi 83].

Our algorithm for computing the Voronoi diagram uses the divide and conquer strategy of [Lee 80, Chew and Drysdale 85] in which left and right Voronoi diagrams are merged. Our major modifications consist of a simplifying choice of the partition line anlge and a fast new $O(N)$ time procedure to calculate the upper end point of the bounded chain. We give a "forbidden region" theorem for all convex metrics that restrains the bisector for a pair of points. This derives monotonicity properties that help our algorithm analysis. It also suggests schemes for numerical approximation of the bisector curves with accuracy independent of the metric.

---

* Supported in part by NSF Grant IRI 8703430 and CCR 8805782.

† Supported in part by NSF Grant IRI 8703430.