Cutting Polygons to Achieve Separability, with Dynamization

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Abstract

Let $P=\{p_1, ..., p_m\}$ be a set of m disjoint simple polygons in the plane with a total of n vertices. A polygon p_i is *separable* from P in a direction d if p_i can be translated to infinity in direction d without colliding with any other polygon p_j of P. Set P is *separable* in direction d, if there is a permutation $(p_{a(1)}, p_{a(2)}, \dots, p_{a(m)})$ such that $p_{a(i)}$ is separable from $\{p_{a(i+1)}, ..., p_{a(m)}\}$, for i=1,...,m-1. In this paper we further investigate the separability problem. We consider the case that other circumstances require that the polygons be sequentially separated in only one given direction d. For example, consider a crane that has to lift a set of objects, one object at a time, out of a deep shaft. If separability is not possible in direction d, the polygons have to be cut in order to allow them to be separated. The process of *cutting* a polygon p_i is represented by introducing a new edge, parallel to direction d, that splits p_i into two disjoint polygons. We investigate the problem of determining a *minimum set of cuts* that achieve separability in direction d.

Our approach is based on a new geometric construction, the *obstruction multigraph* of P in direction d, denoted by $G_d(P)$. This multigraph is defined as follows: The vertices of $G_d(P)$ are the polygons of P, plus two special vertices, s and t, which are associated with two fictitious polygons p_s and p_t at plus infinity and minus infinity in direction d, respectively. (Any polygon of P would collide with either p_s or p_t when moved in direction -d or +d by an arbitrary distance.) The edges incident to a vertex p_i are obtained by following the boundary of polygon p_i in clockwise direction. During the traversal, we add an outgoing edge from p_i to another polygon p_j whenever polygon p_i sees a new polygon p_j in direction d from the current position; similarly, we add an incoming edge from another polygon p_k to p_i whenever polygon p_i is seen by a new polygon p_k in direction d, at the current position. $G_d(P)$ is a directed, planar multigraph with O(n) edges and has exactly one source and one sink; it can be constructed in O(n log n) time.

For every node p_i of $G_d(P)$ representing a polygon p_i of P, let $E(p_i)=(e_1, ..., e_k)$ be the circular sequence of incoming and outgoing edges of p_i. Furthermore, let $ZO(p_i)=(z_1, ..., z_k)$ be the *0-1-sequence* of p_i, where $z_j=1$ if e_j is an incoming edge and $z_j=0$ if e_j is an outgoing edge $(1 \le j< \le k)$. A node p_i of $G_d(P)$ is called *bitonic* if and only if its 0-1-sequence $ZO(p_i)$ is bitonic. A fundamental property of $G_d(P)$ is that the separabilty of P in a direction d can be verified locally at each vertex. We show that P is separable in direction d if and only if every vertex of $G_d(P)$ is bitonic.

The obstruction multigraph allows a minimum set of cuts to be determined efficiently. We show that the minimum number of cuts of polygons of P necessary to achieve separability in direction d is equal to the minimum number of splits necessary to transform the 0-1-sequences of non-bitonic vertices of $G_d(P)$ into bitonic sequences; any minimum cut set for the polygons is associated with a minimum set of cuts of non-bitonic 0-1 sequences (but not every cut of a 0-1-sequence corresponds to a feasable polygon cut). A minimum cut set can be computed in O(n log log n) time using the internal visibility graph (see Tarjan and Van Wyk's algorithm) of each polygon in direction d. Summarizing, we obtain that a minimum set of cuts of polygons of P that achieve separability in direction d can be found in O(n log n) time using O(n) space.

The fact that separability and a minimum cut set can be determined locally at every node of the obstruction multigraph allows an efficient dynamization for decompositions and small translations of individual polygons (without collisions). Clearly, only translations orthogonal to the direction d affect the separability of the the polygon set. An elementary translation of a polygon p_i with respect to direction d is a translation of p_i that alters at most O(1) edges of the obstruction graph $G_d(P)$. Besides elementary translations, we also consider the decomposition of a polygon by means of a vertical cut, or gluing two polygons that share some vertical portion of the boundary. We show that the obstruction graph $G_d(P)$ and the minimum number of polygon cuts that achieve separability in direction d can be updated in time O(log n) after an elementary translation or cut/glue operation.