CHARACTERIZATION OF METRICITY PRESERVING TRANSFORMS

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Abstract

Let A be any set, finite or infinite, and let $d : A \times A \longrightarrow R^+$ be a metric (that is, d is total, positive-definite, symmetric and triangular) on A, where R^+ is the set of non-negative real numbers. If $f: R^+ \longrightarrow R^+$ is a transformation function then we are often faced with the question whether $D : A \times A \longrightarrow R^+$ where $D(x,y) = f(d(x,y))$, $\forall\ x,y \in A$ is a metric on A or not. Examples are the approximations like $\lceil e \rceil$, $\lfloor e \rfloor$, round(e) and e^2 and the normalized functions like $(1-\exp(-e))$ and $m_1 e/(m_2+m_3 e)$, $m_1, m_2, m_3 > 0$, where e is the Euclidean distance between two points in the plane. In general the issue of the metricity of D can be settled if and only if both f and d are known. Often this involves much unnecessary algebraic manipulations which can be easily avoided if certain properties of the transformation f are known. In this paper we provide a simple characterization for the class of those transforms f which produce a metric D for <u>any</u> given metric d. We call them Metricity Preserving Transforms (MPT).

It may be noted, however, that if f is not metricity preserving then we cannot say anything regarding the metricity of D unless d is known, in the sense that depending on the choice of d, f(d) may be a metric. It merely ensures that there exists <u>at least</u> one metric d on some A (e.g., e on the real plane) for which D is not a metric We prove the following to characterize the MPT's.

A transformation $f : R^+ \longrightarrow R^+$ is metricity preserving if and only if (i) f is total, (ii) $f(x) = 0$ if and only if $x = 0$ and (iii) $f(x) + f(y) \geq \max\ \{f(z) \mid |x-y| \leq z \leq x+y\};\ \forall\ x,y \in R^+$.

Widely used MPT's include $f(x) = 1-\exp(-mx)$, $m>0$; $f(x) = m_1 x/(m_2+m_3 x)$, $m_1, m_2, m_3 > 0$; $f(x) = \lceil x \rceil$, $f(x) = ax^b$, $a>0$ and $0<b\leq 1$ and $f(x) = \ln(1+x)$. On the other hand $f(x) = \lfloor x \rfloor$, round(x) $= \lfloor x+0.5 \rfloor$, $f(x) = x^2$, $f(x) = ax^b$, $a>0$ and $b \geq 1$ etc. are not metricity preserving.

MPT's can be combined to form new MPT's too. So if g and h are MPT's then $g \circ h$, $g + h$ and $\max(g,h)$ are also MPT's.

Finally, the concept of metricity preservation may be generalized to n-ary transformations f_n. f_n is a Generalized Metricity Preserving Transform (GMPT) if $D = f_n(d_1, d_2, .., d_n)$ is a metric for every possible choice of metric d_i's, $1 \leq i \leq n$ on any set A. Clearly the characterization can be extended for n arguments, $f_n(d_1, d_2, .., d_n) = d_1 + d_2 + .. + d_n$, $\max(d_1, d_2, ..., d_n)$, $((m_1 d_1^2 + m_2 d_2^2 + ... + m_n d_n^2)/(m_1 + m_2 + .. + m_n))^{1/2}$, $m_i > 0$, $1 \leq i \leq n$ etc. are some of the well known GMPT's.

MPT's can be suitably used to create new classes of metrics from existing ones. The study of the related neighbourhoods and paths of these transformed distances provide scope for future research.