On the Absence of Local Characterizations of Mininmum Weight Triangulations

David Kirkpatrick

Department of Computer Science, University of British Columbia, Vancouver, B.C., Canada V6T 1W5

Abstract

A *triangulation* of a planar point set P is a maximal set of edges with points in P, no two of which properly intersect. The *weight* of a triangulation T is the sum of the Euclidean lengths of its member edges. A triangulation that realizes the minimum weight among all triangulations of P is called a ***minimum weight triangulation*** (MWT).

The decision problem - does there exist a triangulation of a given point set with total weight at most k - has no known polynomial time bounded algorithm nor is it known to be NP-complete (in fact, depending on the model of computation chosen, it is not even clear that the problem is in NP). We will refer to the associated optimization problem - given a point set P, construct a triangulation of minimum weight - as the *minimum weight triangulation problem* (MWT problem).

One serious obstacle to the discovery of an efficient algorithm for the MWT problem is the apparent lack of nice characterizations of MWTs. Delaunay triangulations (the duals of Voronoi diagrams), in contrast, admit the following characterization: for every convex quadralateral in a Delaunay triangulation the diagonal has been chosen so as to maximize the minimum of the six angles in the two triangles forming the quadralateral. At present, the problem of determining if a given triangulation has minimum weight seems to be just as hard as solving the full MWT problem! In this paper, we present some negative evidence for the existence of nice characterizations of MWTs.

If T is a triangulation of a point set P and T' is a triangulation of a subset $P' \subseteq P$ then we say that T' is a ***sub-triangulation*** of T if $T' \subseteq T$. (T' is a ***strict*** subtriangulation if $T' \subset T$). First observe that non-optimality of triangulations is a hereditary property in the sense that if any subtriangulation of T is non-optimal then T is also non-optimal. We show that the converse of this observation does not hold. In particular, there exist arbitrarily large non-optimal triangulations each of whose strict subtriangulations is optimal. In fact, there are convex point sets that admit such triangulations. Thus, the obvious analogs of the Delaunay triangulation characterization, and in fact any characterizations based on local properties of a triangulation, do not hold for MWTs.

Curiously, we demonstrate ***convex*** point sets with the above property. (Straightforward polynomial-time algorithms for the MWT of such point sets are well known.) Our proofs are based on some results concerning the optimal triangulation of regular polygons, which are of interest in their own right.