Divide-and-Conquer in Early Algebraic Topology: The Mayer-Vietoris Exact Homology Sequence Revisited

Alan Saalfeld

Bureau of the Census, Washington, DC, 20233, USA

Abstract

The *Mayer-Vietoris Exact Homology Sequence* relates the homology groups of two topological spaces to the homology groups of the intersection and union of the spaces, provided the spaces are well-behaved. Early algebraic topologists used this sequence to construct more complex topological objects by gluing two spaces together. We will use the sequence to take spaces apart (divide-and-conquer).

$$\cdots \longrightarrow H_i(A \cap B) \longrightarrow H_i(A) \oplus H_i(B) \longrightarrow H_i(A \cup B) \xrightarrow{\partial_i} H_{i-1}(A \cap B) \longrightarrow \cdots$$

This paper examines the proofs and constructions used with Mayer-Vietoris sequences, then extracts information on the combinatorial complexity of the union of geometrically interesting, topologically elementary spaces (finite planar graphs) from the complexity of the original spaces and their intersection. We also observe that the homology exact sequence encodes nice duality properties of the spaces.

As an illustration of an application in automated cartography we show that the number of elementary connected regions arising from overlaying two or more map layers may be computed directly from the connectivity of the line graphs of the two (or more) layers and from the connectivity of the intersection graph(s) of those line graphs. We derive formulas for that computation using the exactness property of the Mayer-Vietoris sequence.

For any plane graph X, let $r(X)$ be the number of regions of the plane separated by X. Then $r(X)$ is the number of connected components in the planar complement of X; $r(X)$ is also one more than the maximum number of independent cycles in the graph X; and $r(X)$ is easily computed using standard graph traversal techniques on X for counting independent cycles. Let $c(X)$ be the number of connected components of X. If A and B are plane graphs to be overlaid, then $A \cup B$ is the plane graph of the overlay; and we have:

$$r(A \cup B) = r(A) - c(A) + r(B) - c(B) - r(A \cap B) + c(A \cap B) + c(A \cup B)$$

All of the values on the right hand side of the equation can be readily computed using standard graph traversal and line intersection algorithms to obtain the desired value, $r(A \cup B)$, the number of regions after overlaying.

To conclude, we examine ∂_i, the boundary function of the induced Mayer-Vietoris exact sequence, in order to specify a generating set for the $r(A \cup B)$ regions in terms of the line intersections among the $c(A \cap B)$ connected components. We show how this generating set may be used to facilitate labeling or coloring the regions.