Combinatorial Face Enumeration in Arrangements and Oriented Matroids

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Abstract

For an arrangement A of n hyperplanes in the projective space P^d or a linear sphere system A of n hyperspheres in the unit sphere S^d, one cannot explicitly give a simple formula for the number $f_k(A)$ of k-dimensional faces (polyhedral regions) because the face numbers depend on the underlying matroid structure. However, as it has been recently shown by the authors, the following relation among the face numbers is valid: $f_k(A) \leq \binom{d}{k} f_d(A)$ for $0 \leq k \leq d$.

In this paper we first show that the same relation is still valid in a more general setting of arrangements topological hyperspheres. This arrangements, known as *sphere systems,* are combinatorially equivalent to oriented matroids. Using the above result, we obtain a polynomial algorithm to enumerate all location $(+,0,-)$-vectors of faces from the set of all location vectors of maximal faces of some sphere system or oriented matroid. This algorithm can be applied to any arrangement of hyperplanes in P^d or in Euclidean space E^d. Combining this with a recent result of Cordovil and Fukuda, we have the following: given the dual graph of an arrangement (where the vertices are the d-faces and two vertices are adjacent if they intersect in a $(d-1)$-face), one can reconstruct the location vectors of all faces of the arrangement up to isomorphism in a polynominal time. It is also shown that one can test in a polynomial time whether a given set of $(+,0,-)$-vectors is the set of maximal vectors (topes) of an oriented matroid.

For clearness and rigorousness of proofs and constructions, we depend strongly on both simple axiomatizations of oriented matroids and inductive arguments through oriented matroid minors.