Convex Hulls of Points from Spherically Symmetric Distributions

Rex A. Dwyer

Department of Computer Science, Box 8206, North Carolina State University, Raleigh, North Carolina, 27695–8206

Abstract

This work investigates the expected combinatorial complexity of the convex hull of n independent and identically distributed points in $\mathbf{R}^d$. In particular, it derives asymptotic bounds on EV_n, the expected number of vertices; EF_n, the expected number of facets; and ET_n, the expected running time for convex-hull construction. In the worst case, $V_n = n$ and $F_n = \Theta(n^{\lfloor d/2 \rfloor})$; however, for many distributions much smaller bounds are known for EV_n, EF_n, and ET_n. Others have investigated many particular distributions; this work extends to higher dimensions Carnal's results on convex hulls of samples from three broad classes of circularly symmetric distributions in the plane. (H. Carnal, "Die konvexe Hülle von n rotationssymmetrisch verteilten Punkten", *Z. Wahrscheinkeitstheorie verw. Geb.* 15, 168–176(1970).) A further result relates to distributions uniform on the Cartesian product of balls of various dimensions.

A density function f on $\mathbf{R}^d$ is *spherically symmetric* if $f(x) = f(y)$ whenever $\|x\| = \|y\|$. Let $L(x)$ be *slowly varying.* (Loosely, $L(x) = o(n^\alpha)$ for all positive α.) Let $F(x) = \Pr\{\|X\| \geq x\}$.

Theorem 1 (Algebraic tails) *For distributions satisfying $F(x) = x^{-k}L(x)$ with $k \geq 0$, $EV_n = \Theta(1)$, $EF_n = \Theta(1)$, and $ET_n = \Theta(n)$.*

Theorem 2 (Exponential tails) *For distributions satisfying $x = L(1/F(x))$ with $L(x)$ satisfying certain technical smoothness conditions, EV_n and EF_n are slowly varying, and $ET_n = \Theta(n)$. In fact, $EV_n = \Theta\left(\frac{L(n)}{nL'(n)}\right)^{(d-1)/2}$ and $EF_n = O\left(\frac{L(n)}{nL'(n)}\right)^{\lfloor d/2 \rfloor (d-1)/2}$.*

Theorem 3 (Truncated tails) *For distributions in the unit d-ball satisfying $F(1-x) \sim cx^k$ for positive k, $EV_n = \Theta(n^{(d-1)/(2k+d-1)})$ and $EF_n = \Theta(n^{(d-1)/(2k+d-1)})$. In every case, $ET_n = o(n^2)$; if $k > (d-1)/2$, then $ET_n = \Theta(n)$.*

Theorem 4 *Let $\mathcal{B}$ be the Cartesian product of k balls of dimensions $d_1, d_2, \ldots, d_k$, with $d_1 \geq d_2 \geq \ldots \geq d_k$. Let m be the largest i for which $d_i = d_1$, that is, the number of balls of the largest dimension. For the uniform distribution on $\mathcal{B}$, $EV_n = \Theta(n^{(d_1-1)/(d_1+1)} \log^{m-1} n)$.*