The Complexity of Order Types—A Survey

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Abstract

The notion of the "order type" of a finite configuration of points is the natural generalization to two or more dimensions of the order of points on a line. In this survey talk we discuss "order type" as a method of classifying point configurations, and relate it to the question of the combinatorial complexity of a configuration. We will focus on three main issues: (a) the λ-matrix of a configuration, which (by very recent results) gives the most compact known encoding of its order type; (b) the isotopy problem for configurations, whose solution by Mnev has just become generally known; and (c) the recent solution by Goodman, Pollack, and Sturmfels of Chazelle's problem of the integral realization of order types, and its implications for the issue of complexity.

(a) In a 1983 paper entitled Multidimensional Sorting, Goodman and Pollack proved that the order type of a d-dimensional configuration S, i.e., the set of orientations of all its $(d+1)$-tuples, is uniquely determined by the knowledge of how many points of S lie on the positive side of each oriented hyperplane spanned by d independent points of S. This result, which we called the "basic theorem of geometric sorting", implied that the order type of a configuration of n points in $\mathbf{R}^d$, which on the face of it has complexity $\Omega(n^{d+1})$, can be encoded in space $O(n^d)$ (i.e., using at most $n^d \log n$ bits); an algorithm was described in the same paper for carrying out this d-dimensional sorting, of time complexity $O(n^d \log n)$, which was subsequently improved to an optimal $O(n^d)$ algorithm by Edelsbrunner, O'Rourke, and Seidel. Because the λ-matrix also encodes non-realizable oriented matroids (i.e., "pseudo-configurations"), in addition to realizable ones, and because of the result of Goodman and Pollack, published in 1986, that most (in a very strong sense) oriented matroids are not realizable, it was felt that a more space-efficient encoding of realizable order types should be possible than that given by the λ-matrix. It was thought, for example, that the naive method of choosing an integral representative of each order type might well provide such a more compact encoding. As we shall see, however, this has now turned out not to be the case.

(b) An outstanding problem for a number of years has been the so-called isotopy problem for point configurations: if two configurations, in general position, say, have the same order type, can one be continuously deformed into the other within that order type? In the plane, for example, if points $P_1, \ldots, P_n$ are in general position and $Q_1, \ldots, Q_n$ have the same order type, can we move the P's to the Q's so that at no time do three of the points line up? Attempts to prove this have succeeded only for small values of n, and for good reason as it turns out: several years ago the Soviet mathematician N. E. Mnev, a student of A. M. Vershik, proved not only that this is false, i.e., that the space of realizations of a general position order type may be disconnected, but that this space may have the same homotopy type as *any* semi-algebraic set, a much more sweeping result. (Independently, Beat Jaggi and Peter Mani-Levitska came up with a fairly small counterexample, involving 17 points, and it is not yet clear just how small n can be made for a planar counterexample.) Related results hold for polytopes in dimensions greater than 3.

(c) At the first ACM Symposium on Computational Geometry, Bernard Chazelle asked for bounds on an integer N such that every realizable order type of n points could be realized by integer lattice points in the grid $|x|, |y| \leq N$. (His question, repeated a couple of years later at a Computational Geometry Day at the Courant Institute, was actually about realizations by lines having integer coefficients, but one question can easily be reduced to the other.) Recently Goodman, Pollack, and Sturmfels were able to establish doubly-exponential lower and upper bounds for this grid-size, the lower bound with the help of a construction due to Sturmfels and White for passing from a degenerate order type to a nondegenerate one, the upper bound making use of some recent results of Grigor'ev and Vorobjov on solutions of simultaneous inequalities. In particular, the lower bound shows that exponential storage is necessary in the worst case for the encoding of an order type by an integral representative, so that the problem of improving the λ-matrix encoding still remains open.