Flips in Edge-Labelled Pseudo-Triangulations*

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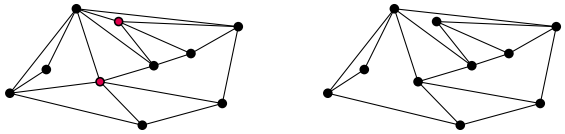
Abstract

We show that $O(n^2)$ exchanging flips suffice to transform any edge-labelled pointed pseudo-triangulation into any other with the same set of labels. By using insertion, deletion and exchanging flips, we can transform any edge-labelled pseudo-triangulation into any other with $O(n \log c + h \log h)$ flips, where c is the number of convex layers and h is the number of points on the convex hull.

1 Introduction

A *pseudo-triangle* is a simple polygon with three convex interior angles, called *corners*, that are connected by reflex chains. Given a set P of n points in the plane, a *pseudo-triangulation* of P is a subdivision of its convex hull into pseudo-triangles, using all points of P as vertices (see Figure 1a). A pseudo-triangulation is *pointed* if all vertices are incident to a reflex angle in some face (including the outer face; see Figure 1b for an example). Pseudo-triangulations find applications in areas such as kinetic data structures [6] and rigidity theory [9]. More information on pseudo-triangulations can be found in a survey by Rote, Santos, and Streinu [8].



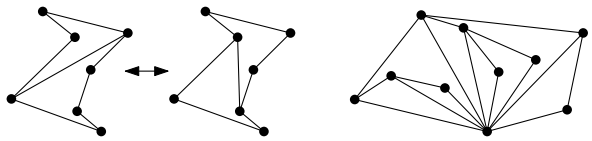
(a) (b)

Figure 1: (a) A pseudo-triangulation with two non-pointed vertices. (b) A pointed pseudo-triangulation.

Since a regular triangle is also a pseudo-triangle, pseudo-triangulations generalize triangulations (subdivisions of the convex hull into triangles). Triangulations have numerous applications and are extremely well-studied; the particular topic we are interested in is that of flips. In a triangulation, a flip is a local transformation that removes one edge, leaving an empty quadrilateral, and inserts the other diagonal of that quadrilateral. Note that this is only possible if the quadrilateral is convex. Lawson [7] showed that any triangulation with n vertices can be transformed into any other with $O(n^2)$ flips, and Hurtado, Noy, and Urrutia [5] gave a matching $\Omega(n^2)$ lower bound.

Pointed pseudo-triangulations support a similar type of flip, but before we can introduce this, we need to generalize the concept of pseudo-triangles to *pseudo-k-gons*: weakly simple polygons with k convex interior angles. A diagonal of a pseudo-k-gon is called a *bitangent* if the pseudo-k-gon remains pointed after insertion of the diagonal. In a pointed pseudo-triangulation, *flipping* an edge removes the edge, leaving a pseudo-quadrilateral (a pseudo-4-gon), and inserts the unique other bitangent of the pseudo-quadrilateral (see Figure 2a). In contrast with triangulations, all internal edges of a pointed pseudo-triangulation are flippable. Bereg [3] showed that $O(n \log n)$ flips suffice to transform any pseudo-triangulation into any other.

Aichholzer et al. [2] showed that the same result holds for all pseudo-triangulations (including triangulations) if we allow two more types of flips: *insertion* and *deletion* flips. As the name implies, these either insert or delete one edge, provided that the result is still a pseudo-triangulation. To disambiguate, they call the other flips *exchanging* flips. In a later paper, this bound was refined to $O(n \log c)$ [1], where c is the number of convex layers of the point set.



(a) (b)

Figure 2: (a) A flip in a pseudo-quadrilateral. (b) A left-shelling pseudo-triangulation.

In this paper, we investigate flips in *edge-labelled pseudo-triangulations*: pseudo-triangulations where each internal edge has a unique label in $\{1, \ldots, 3n-3-2h\}$, where h is the number of vertices on the convex hull ($3n-3-2h$ is the number of internal edges in a triangulation). In the case of an exchanging flip, the new edge receives the label of the old edge. For a deletion flip, the edge and its label are simply removed, and for an insertion flip, the new edge receives an unused label from the set of all possible labels. The edge-labelled version is more difficult than the unlabelled version, since we no longer have the freedom to choose the mapping between

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edges in the initial and final pseudo-triangulation.

Bose et al. [4] initiated the study of flips in edge-labelled triangulations. They gave a tight $\Theta(n \log n)$ bound on the worst-case number of flips required for triangulations of points in convex position. However, in general, they show that it is not always possible to flip between two given edge-labelled triangulations. In contrast, we show that it is always possible to transform two given edge-labelled pseudo-triangulations into each other using flips.

Our results are the following: using only exchanging flips, we show that $O(n^2)$ flips suffice to transform any edge-labelled pointed pseudo-triangulation into any other with the same set of labels. By using insertion, deletion and exchanging flips, we can transform any edge-labelled pseudo-triangulation into any other with $O(n \log c + h \log h)$ flips.

Before we prove our results, we need a few more definitions. Given a set of points in the plane, let v_0 be the point with the lowest y-coordinate, and let $v_1, \ldots, v_n$ be the other points in clockwise order around v_0. The *left-shelling* pseudo-triangulation is the union of the convex hulls of $v_0, \ldots, v_i$, for all $2 \leq i \leq n$ (see Figure 2b). Thus, every vertex after v_1 is associated with two edges: a *bottom* edge connecting it to v_0 and a *top* edge that is tangent to the convex hull of the earlier vertices. The *right-shelling* pseudo-triangulation is similar, with the vertices added in counter-clockwise order instead.

2 Transforming pointed pseudo-triangulations

In this section, we show that every edge-labelled pointed pseudo-triangulation can be transformed into any other with the same set of labels by $O(n^2)$ exchanging flips. We do this by showing how to transform a given edge-labelled pointed pseudo-triangulation into a *canonical* one. The result then follows by the reversibility of flips. We use the left-shelling pseudo-triangulation as canonical pseudo-triangulation, with the bottom edges labelled in clockwise order around v_0, followed by the internal top edges in the same order (based on their associated vertex).

Since we can transform any pointed pseudo-triangulation into the left-shelling pseudo-triangulation with $O(n \log n)$ flips [3], the main part of the proof lies in reordering the labels of a left-shelling pseudo-triangulation. We use two tools for this, called a *sweep* and a *shuffle*, that are implemented by a sequence of flips. A sweep interchanges the labels of some internal top edges with their respective bottom edges, while a shuffle permutes the labels on all bottom edges.

Lemma 1 *We can transform any left-shelling pseudo-triangulation into the canonical one with $O(1)$ shuffle and sweep operations.*

Proof. In the canonical pseudo-triangulation, we call the labels assigned to bottom edges *low*, and the labels assigned to top edges *high*. In the first step, we use a shuffle to line up every bottom edge with a high label to a top edge with a low label. Then we exchange these pairs of labels with a sweep. Now all bottom edges have low labels and all top edges have high labels, so all that is left is to sort the labels. We can sort the low labels with a second shuffle. To sort the high labels, we sweep them to the bottom edges, shuffle to sort them there, then sweep them back. □

The remainder of this section describes how to perform a sweep and a shuffle with flips.

Lemma 2 *We can interchange the labels of the edges incident to an internal vertex v of degree two with three exchanging flips.*

Proof. Consider what happens when we remove v. Deleting one of its edges leaves a pseudo-quadrilateral. Removing the second edge then either merges two corners into one, or removes one corner, leaving a pseudo-triangle T. There are three bitangents that connect v to T, each corresponding to the geodesic between v and a corner of T. Any choice of two of these bitangents results in a pointed pseudo-triangulation. When one of them is flipped, the only new edge that can be inserted so that the result is still a pointed pseudo-triangulation is the bitangent that was not there before the flip. Thus, we can interchange the labels with three flips (see Figure 3). □

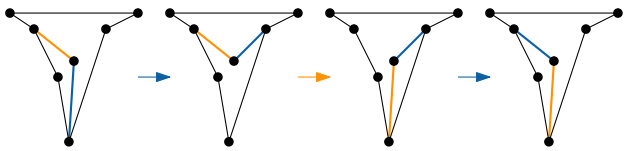


Figure 3: Interchanging the labels of the edges incident to a vertex of degree two.

Lemma 3 (Sweep) *In the left-shelling pseudo-triangulation, we can interchange the labels of any number of internal top edges and their corresponding bottom edges with $O(n)$ exchanging flips.*

Proof. Let S be the set of vertices whose internal top edge should have its label swapped with the corresponding bottom edge. Consider a ray L from v_0 that starts at the positive x-axis and sweeps through the point set to the negative x-axis. We will maintain the following invariant: the graph induced by the vertices to the left of L is their left-shelling pseudo-triangulation and the graph induced by the vertices to the right of L is their right-shelling pseudo-triangulation (both groups include

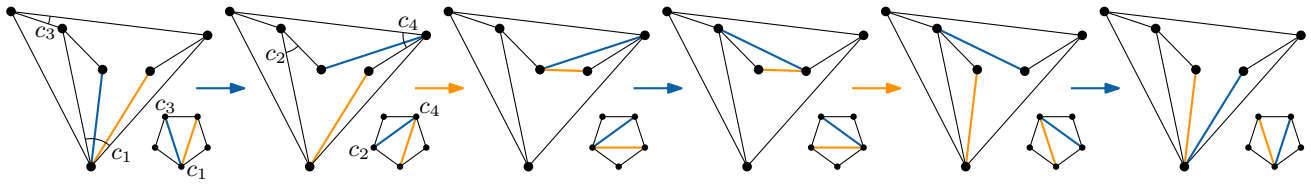


Figure 4: Interchanging the labels of two bitangents of a pseudo-pentagon with five bitangents. An edge in the pentagon corresponds to a geodesic between two corners of the pseudo-pentagon.

v_0). Furthermore, the labels of the top edges of the vertices in S to the right of L have been interchanged with their respective bottom edges. This invariant is satisfied at the start.

Suppose that L is about to pass a vertex v_k. If v_k is on the convex hull, its top edge is not internal and no action is required for the invariant to hold after passing v_k. So assume that v_k is not on the convex hull and consider its incident edges. It is currently part of the left-shelling pseudo-triangulation of points to the left of L, where it is the last vertex. Thus, v_k is connected to v_0 and to one vertex to its left. It is not connected to any vertex to its right, since there are $2n-3$ edges in total, and the left- and right-shelling pseudo-triangulations to each side of L contribute $2(k+1)-3+2(n-k)-3 = 2n-4$ edges. So the only edge that crosses L is an edge of the convex hull. Therefore v_k has degree two, which means that we can use Lemma 2 to swap the labels of its top and bottom edge with three flips if $v_k \in S$.

Furthermore, the sides of the pseudo-triangle that remains if we were to remove v_k, form part of the convex hull of the points to either side of L. Thus, flipping the top edge of v_k results in the tangent from v_k to the convex hull of the points to the right of L – exactly the edge needed to add v_k to their right-shelling pseudo-triangulation. Therefore we only need $O(1)$ flips to maintain the invariant when passing v_k.

At the end, we have constructed the right-shelling pseudo-triangulation and swapped the desired edges. An analogous transformation without any swapping can transform the graph back into the left-shelling pseudo-triangulation with $O(n)$ flips in total. □

Lemma 4 *In the left-shelling pseudo-triangulation, we can interchange the labels of two consecutive bottom edges with $O(1)$ exchanging flips.*

Proof. When we remove the two consecutive bottom edges (say a and b), we are left with a pseudo-pentagon X. A pseudo-pentagon can have up to five bitangents, as each bitangent corresponds to a geodesic between two corners. If X has exactly five bitangents, this correspondence is a bijection. This implies that the bitangents of X can be swapped just like diagonals of a convex pentagon (see Figure 4). On the other hand, if X has only four bitangents, it is impossible to swap a and b without flipping an edge of X.

Fortunately, we can always transform X into a pseudo-pentagon with five bitangents. If the pseudo-triangle to the right of b is a triangle, X already has five bitangents (see Lemma 19 in the Appendix). Otherwise, the top endpoint of b is an internal vertex of degree two and we can flip its top edge to obtain a new pseudo-pentagon that does have five bitangents (see Lemma 20 in the Appendix). After swapping the labels of a and b, we can flip this top edge back. Thus, in either case we can interchange the labels of a and b with $O(1)$ flips. □

We can use Lemma 4 to reorder the labels of the bottom edges with insertion or bubble sort, as these algorithms only swap adjacent values.

Corollary 5 (Shuffle) *In the left-shelling pseudo-triangulation, we can reorder the labels of all bottom edges with $O(n^2)$ exchanging flips.*

Combining this with Lemmas 1 and 3, and the fact that we can transform any pointed pseudo-triangulation into the left-shelling one with $O(n \log n)$ flips [3], gives the main result.

Theorem 6 *We can transform any edge-labelled pointed pseudo-triangulation with n vertices into any other with $O(n^2)$ exchanging flips.*

The following lower bound follows from the $\Omega(n \log n)$ lower bound on the flip distance between edge-labelled triangulations of a convex polygon [4].

Theorem 7 *There are pairs of edge-labelled pointed pseudo-triangulations with n vertices that require $\Omega(n \log n)$ exchanging flips to transform one into the other.*

3 Transforming general pseudo-triangulations

In this section, we extend our results for edge-labelled pointed pseudo-triangulations to all edge-labelled pseudo-triangulations. Since not all pseudo-triangulations have the same number of edges, we need to allow flips that change the number of edges. In particular, we allow a single edge to be deleted or inserted, provided that the result is still a pseudo-triangulation.

Since we are dealing with edge-labelled pseudo-triangulations, we need to determine what happens to the edge labels. It is useful to first review the properties we would like these flips to have. First, a flip should be a local operation – it should affect only one edge. Second, a labelled edge should be flippable if and only if the edge is flippable in the unlabelled setting. This allows us to re-use the existing results on flips in pseudo-triangulations. Third, flips should be reversible. Like most proofs about flips, our proof in the previous section crucially relies on the reversibility of flips.

With these properties in mind, the edge-deletion flip is rather straightforward – the labelled edge is removed, and other edges are not affected. Since the edge-insertion flip needs to be the inverse of this, it should insert the edge and assign it a *free label* – an unused label in $\{1, \ldots, 3n-3-2h\}$, where h is the number of vertices on the convex hull ($3n-3-2h$ is the number of internal edges in a triangulation).

With the definitions out of the way, we turn our attention to the number of flips required to transform any edge-labelled pseudo-triangulation into any other. In this section, we show that by using insertion and deletion flips, we can shuffle (permute the labels on bottom edges) with $O(n + h \log h)$ flips. Combined with the unlabelled bound of $O(n \log c)$ flips by Aichholzer et al. [1], this brings the total number of flips down to $O(n \log c + h \log h)$. Note that, by the results of Bose et al. [4], this holds for a set of points in convex position ($h = n$). In the remainder of this section we assume that $h < n$. As before, we first build a collection of simple tools that help prove the main result.

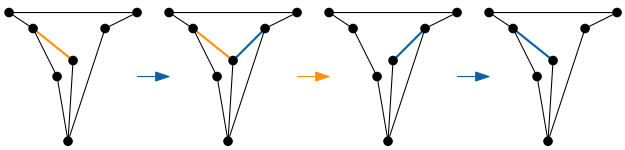


Figure 5: Interchanging the label of an edge incident to a vertex of degree two with a free label.

Lemma 8 *With $O(1)$ flips, we can interchange the label of an edge incident to an internal vertex of degree two with a free label.*

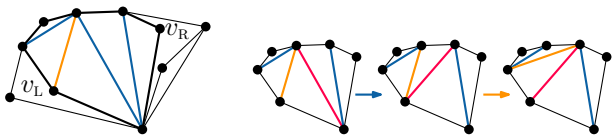
Proof. Let v be a vertex of degree two and let e be an edge incident to v. Since v has degree two, its removal leaves an empty pseudo-triangle T. There are three bitangents that connect v to T, one for each corner. Thus, we can insert the third bitangent f with the desired free label, making v non-pointed (see Figure 5). Flipping e now removes it and frees its label. Finally, flipping f moves it into e's starting position, completing the exchange. □

This implies that, using an arbitrary free label as placeholder, we can swap any two edges incident to internal degree-two vertices – no matter where they are in the pseudo-triangulation.

Corollary 9 *We can interchange the labels of two edges, each incident to some internal vertex of degree two, with $O(1)$ flips.*

Recall that during a sweep (Lemma 3), each internal vertex has degree two at some point. Since the number of free labels for a pointed pseudo-triangulation is equal to the number of internal vertices, this means that we can use Lemma 8 to swap every label on a bottom edge incident to an internal vertex with a free label by performing a single sweep. Afterwards, a second sweep can replace these labels on the bottom edges in any desired order. Thus, permuting the labels on bottom edges incident to internal vertices can be done with $O(n)$ flips. Therefore, the difficulty in permuting the labels on all bottom edges lies in bottom edges that are not incident to an internal vertex, that is, chords of the convex hull. If there are few such chords, a similar strategy (free them all and replace them in the desired order) might work. Unfortunately, the number of free labels can be far less than the number of chords.

We now consider operations on maximal groups of consecutive chords, which we call *fans*. As the vertices of a fan are in convex position, fans behave in many ways like triangulations of a convex polygon, which can be rearranged with $O(n \log n)$ flips [4]. The problem now becomes getting the right set of labels on the edges of a fan.



(a) (b)

Figure 6: (a) An indexed fan. (b) Shifting the index.

Consider the internal vertices directly to the left (v_L) and right (v_R) of a fan F, supposing both exist. Vertex v_L has degree two and forms part of the reflex chain of the first pseudo-triangle to the left of F. Thus, flipping v_L's top edge connects it to the leftmost vertex of F (excluding v_0). Vertex v_R is already connected to the rightmost vertex of F, so we just ensure that it has degree two. To do this, we flip all incident edges from vertices further to the right, from the bottom to the top. Now the diagonals of F form a triangulation of a convex polygon whose boundary consists of v_0, v_L, the top endpoints of the chords, and v_R (see Figure 6a). It is possible that there is no internal vertex to one side of F. In that case, there is only one vertex on that side of

F, which is part of the convex hull, and we can simply use that vertex in place of v_{L} or v_{R} without flipping any of its edges. Since there is at least one internal vertex by assumption, either v_{L} or v_{R} is an internal vertex. This vertex is called the *index* of F. If a vertex is the index of two fans, it is called a *shared index*.

A triangulated fan is called an *indexed fan* if there is one edge incident to the index, the *indexed edge*, and the remaining edges are incident to one of the neighbours of the index on the boundary. Initially, all diagonals of F are incident to v_0, so we transform it into an indexed fan by flipping the diagonal of F closest to the index. Next, we investigate several operations on indexed fans that help us move labels between fans.

Lemma 10 (Shift) *In an indexed fan, we can shift the indexed edge to the next diagonal with $O(1)$ flips.*

Proof. Suppose that v_{L} is the index (the proof for v_{R} is analogous). Let e be the current indexed edge, and f be the leftmost diagonal incident to v_0. Then flipping f followed by e makes f the only edge incident to the index and e incident to the neighbour of the index (see Figure 6b). Since flips are reversible, we can shift the index the other way too. □

Lemma 11 *We can switch which fan a shared index currently indexes with $O(1)$ flips.*

Proof. Flipping the current indexed edge "parks" it by connecting it to the two neighbours of the index, and reduces the degree of the index to two (see Figure 7). Now, flipping the top edge of the index connects it to the other fan, where we parked the previously indexed edge. Flipping that edge connects it to the index again. □

Lemma 12 *In a pointed pseudo-triangulation, we can always decrease the degree of a vertex v of degree three by flipping one of the edges incident to its reflex angle.*

Proof. Consider the geodesic from v to the opposite corner c of the pseudo-triangle v is pointed in. The line supporting the part of the geodesic when it reaches v splits the edges incident to v into two groups. As there are three edges, one of these groups must contain multiple edges. Flipping the edge incident to its reflex angle in the group with multiple edges results in a geodesic to c. If this geodesic passed through v, it would insert the missing edges along the geodesic from v to c (otherwise we could find a shorter path). But inserting this geodesic would make v non-pointed. Thus, v cannot be on this geodesic. Therefore the new edge is not incident to v and the flip reduces the degree of v. □

Since the index always has degree three, this allows us to extend the results from Lemma 8 and Corollary 9 regarding vertices of degree two to indexed edges.

Corollary 13 *In an indexed fan, we can interchange the label of the indexed edge with a free label in $O(1)$ flips.*

Corollary 14 *Given two indexed fans, we can interchange the labels of the two indexed edges with $O(1)$ flips.*

Now we have enough tools to shuffle the bottom edges.

Lemma 15 (Shuffle) *In the left-shelling pseudo-triangulation, we can reorder the labels of all bottom edges with $O(n + h \log h)$ flips, where h is the number of vertices on the convex hull.*

Proof. In the initial pseudo-triangulation, let B and $\mathcal{F}$ be the sets of labels on bottom edges and free labels, respectively. Let F_i be the set of labels on the i-th fan (in some fixed order), and let $\overline{F}$ be the set of labels on non-fan bottom edges. Let F_i' and $\overline{F}'$ be these same sets in the target pseudo-triangulation. As we are only rearranging the bottom labels, we have that $B = F_1 \cup \ldots \cup F_k \cup \overline{F} = F_1' \cup \ldots \cup F_k' \cup \overline{F}'$, where k is the number of fans.

We say that a label ℓ *belongs to* fan i if $\ell \in F_i'$. At a high level, the reordering proceeds in four stages. In stage one, we free all labels in $\overline{F}$. In stage two, we place each label from $B \setminus \overline{F}'$ in the fan it belongs to, leaving the labels in $\overline{F}'$ free. Then, in stage three, we correct the order of the labels within each fan. Finally, we place the labels in $\overline{F}'$ correctly.

Since each internal vertex contributes exactly one top edge, one bottom edge, and one free label, we have that $|\overline{F}| = |\mathcal{F}|$. To free all labels in $\overline{F}$, we perform a sweep (see Lemma 3). As every internal vertex has degree two at some point during the sweep, we can exchange the label on its bottom edge with a free label at that point, using Lemma 8. This requires $O(n)$ flips. The labels in $\mathcal{F}$ remain on the bottom edges incident to internal vertices throughout stage two and three, as placeholders.

To begin stage two, we index all fans with $O(n)$ flips and shift these indices to the first 'foreign' edge: the first edge whose label does not belong to the current fan. If no such edge exists, we can ignore this fan for the remainder of stage two, as it already has the right set of labels. Now suppose that there is a fan F_i whose indexed edge e is foreign: $\ell_e \notin F_i'$. Then either $\ell_e \in F_j'$ for some $j \neq i$, or $\ell_e \in \overline{F}'$. In the first case, we exchange ℓ_e with the label on the indexed edge of F_j, and shift the index of F_j to the next foreign edge. In the second case, we exchange ℓ_e with a free label in $B \setminus \overline{F}'$. If this label belongs to F_i, we shift its index to the next foreign edge. In either case, we increased the number of correctly placed labels by at least one. Thus $n - 1$ repetitions suffice to place all labels in the

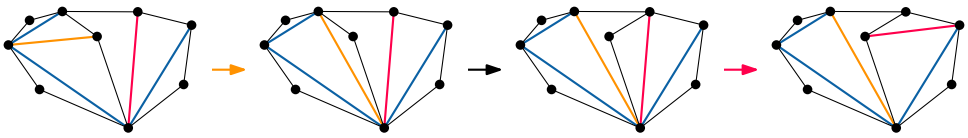


Figure 7: Changing which side a shared index indexes.

fan they belong to, wrapping up stage two. Since we perform a linear number of swaps and shifts, and each takes a constant number of flips, the total number of flips required for stage two is $O(n)$.

For stage three, we note that each indexed fan corresponds to a triangulation of a convex polygon. As such, we can rearrange the labelled diagonals of a fan F_i into their desired final position with $O(|F_i| \log |F_i|)$ flips [4]. Thus, if we let h be the number of vertices on the convex hull, the total number of flips for this step is bounded by

$$\sum_i O(|F_i| \log |F_i|) \leq \sum_i O(|F_i| \log h) = O(h \log h).$$

For stage four, we first return to a left-shelling pseudo-triangulation by un-indexing each fan, using $O(n)$ flips. After stage two, the labels in $\overline{F}'$ are all free, so all that is left is to place these on the correct bottom edges, which we can do with a final sweep. Thus, we can reorder all bottom labels with $O(n + h \log h)$. □

This leads to the following bound.

Theorem 16 *We can transform any edge-labelled pseudo-triangulation with n vertices into any other with $O(n \log c + h \log h)$ flips, where c is the number of convex layers and h is the number of vertices on the convex hull.*

Proof. Using the technique by Aichholzer et al. [1], we first transform the pseudo-triangulation into the left-shelling pseudo-triangulation T with $O(n \log c)$ flips. Our canonical pseudo-triangulation contains the labels $\{1, \ldots, 2n - h - 3\}$, but it is possible for T to contain a different set of labels. Since all labels are drawn from $\{1, \ldots, 3n - 2h - 3\}$, at most $n - h$ labels differ. This is exactly the number of internal vertices. Thus, we can use $O(n + h \log h)$ flips to shuffle (Lemma 15) all non-canonical labels on fan edges to bottom edges incident to an internal vertex. Once there, we use a sweep (Lemma 3) to ensure that every internal vertex has degree two at some point, at which time we replace its incident non-canonical labels with canonical ones with a constant number of flips (Lemma 8). Once our left-shelling pseudo-triangulation has the correct set of labels, we use a constant number of shuffles and sweeps to sort the labels (Lemma 1). Since we can shuffle and sweep with $O(n + h \log h)$ and $O(n)$ flips, respectively, the total number of flips reduces to $O(n \log c + n + h \log h) = O(n \log c + h \log h)$. □

The correspondence between triangulations of a convex polygon and pseudo-triangulations gives us the following lower bound.

Theorem 17 *There are pairs of edge-labelled pseudo-triangulations with n vertices such that any sequence of flips that transforms one into the other has length $\Omega(n \log n)$.*

Acknowledgements

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Appendix

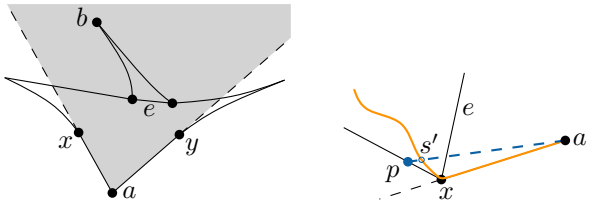

Figure 8: (a) A corner of a pseudo-triangle and an edge such that the entire pseudo-triangle on the other side of the edge lies inside the corner's wedge. (b) If a can see a point past x, then the geodesic does not contain x.

Lemma 18 *Let a be a corner of a pseudo-triangle with neighbours x and y, and let e be an edge on the chain opposite a. If all vertices of the other pseudo-triangle containing e lie in the wedge formed by extending the edges ax and ay into half-lines (see Figure 8a), then flipping e will result in an edge incident on a.*

Proof. Let T be the pseudo-triangle on the other side of e, and let b be the corner of T opposite e. Then flipping e inserts the geodesic between a and b. This geodesic must intersect e in a point s and then follow the shortest path from s to a. If s lies strictly inside the wedge, nothing can block as, thus the new edge will contain as and be incident on a.

Now, if all of e lies strictly inside the wedge, our result follows. But suppose that e has x as an endpoint and the geodesic between a and b intersects e in x. As a can see x and all of T lies inside the wedge, there is an $\varepsilon > 0$ such that a can see the point X on the boundary of T at distance ε from x (see Figure 8b). The line segment ap intersects the geodesic at a point s'. By the triangle inequality, $s'a$ is shorter than following the geodesic from s' via x to a. But then this would give a shorter path between a and b, by following the geodesic to s' and then cutting directly to a. As the geodesic is the shortest path by definition, this is impossible. Thus, the geodesic cannot intersect e at x and the new edge must be incident to a. □

Lemma 19 *Let a and b be two consecutive internal bottom edges in the left-shelling pseudo-triangulation, such that the pseudo-triangle to the right of b is a triangle. Then the pseudo-pentagon X formed by removing a and b has five bitangents.*

Proof. Let $c_0, \ldots, c_4$ be the corners of X in counterclockwise order around the boundary. By Lemma 18, flipping b results in an edge b' that intersects b and is incident on c_1. This edge is part of the geodesic between c_1 and c_3, and as such it is tangent to the convex chain $v_0, v_a, \ldots, c_3$, where v_a is the top endpoint of a (v_a could be c_3). Therefore it is also the tangent from c_1 to the convex hull of $\{v_0, \ldots, v_a\}$. This means that the newly created pseudo-triangle with c_1 as corner and a on the opposite pseudo-edge also meets the conditions of Lemma 18. Thus, flipping a results in another edge, a', also incident on c_1. As b separates c_1 from all vertices in $\{v_0, \ldots, v_a\}$, a' must also intersect b. This gives us four bitangents, of which two are incident on v_0 (a and b), and two on c_1 (a' and b'). Finally, flipping a before flipping b results in a bitangent that is not incident on v_0 (as v_0 is a corner and cannot be on the new geodesic), nor on c_1 (as b separates a from c_1). Thus, X has five bitangents. □

Lemma 20 *Let a and b be two consecutive internal bottom edges in the left-shelling pseudo-triangulation, such that the pseudo-triangle to the right of b is not a triangle. Then the pseudo-pentagon X formed by flipping the corresponding top edge of b and removing a and b has five bitangents.*

Proof. Let v_a and v_b be the top endpoints of a and b. By Lemma 18 and since b had degree two, flipping the top edge of b results in the edge $v_b c_1$. We get three bitangents for free: a, b, and b' – the old top edge of b and the result of flipping b.

X consists of a reflex chain C that is part of the convex hull of the points to the left of a, followed by three successive tangents to C, v_a, or v_b. Since C lies completely to the left of a, it cannot significantly alter any of the geodesics or bitangents inside the polygon, so we can reduce it to a single edge. Now, X consists either of a triangle with two internal vertices, or a convex quadrilateral with one internal vertex.

If X is a triangle with two internal vertices, the internal vertices are v_a and v_b. Let its exterior vertices be v_0, x, and y. Then there are seven possible bitangents: $a = v_0v_a, b = v_0v_b, xv_a, xv_b, yv_a, yv_b$, and v_av_b. We know that xv_a and yv_b are edges, so there are five possible bitangents left. As all vertices involved are either corners or have degree one in X, the only condition for an edge to be a bitangent is that it does not cross the boundary of X. Since the exterior boundary is a triangle, this reduces to it not crossing xv_a and yv_b. Two line segments incident to the same vertex cannot cross. Thus, xv_b, yv_a, and v_av_b cannot cross xv_a and yv_b, and X has five bitangents.

If X's convex hull has four vertices, the internal vertex is v_b (otherwise the pseudo-triangle to the right of b would be a triangle). Let its exterior vertices be v_0, x, v_a, and y. Then there are six possible bitangents: $a = v_0v_a, b = v_0v_b, xy, xv_b, yv_b$, and v_av_b, of which one (yv_b) is an edge of X. Since a and b are guaranteed to be bitangents, and xy, xv_b, and v_av_b all share an endpoint with yv_b, the arguments from the previous case apply and we again have five bitangents. □