The Shadows of a Cycle Cannot All Be Paths

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Abstract

A *shadow* of a subset S of Euclidean space is an orthogonal projection of S into one of the coordinate hyperplanes. In this paper we show that it is not possible for all three shadows of a cycle (i.e., a simple closed curve) in $\mathbb{R}^3$ to be paths (i.e., simple open curves).

We also show two contrasting results: the three shadows of a path in $\mathbb{R}^3$ can all be cycles (although not all convex) and, for every $d \geq 1$, there exists a d-sphere embedded in $\mathbb{R}^{d+2}$ whose $d+2$ shadows have no holes (i.e., they deformation-retract onto a point).

1 Introduction

Oskar's maze, named after the Dutch puzzle designer Oskar van Deventer, who invented it in 1983, is a mechanical puzzle consisting of a hollow cube and three mutually orthogonal rods joined at their centers (see Figure 1). Each face of the cube has slits forming a maze, and the mazes on opposite faces are identical. Each rod is orthogonal to a pair of opposite faces, and it is able to slide in the slits, tracing out the maze. Hence, in order to move the rods around, one has to solve three mazes simultaneously.

In 1994, Hendrik W. Lenstra asked if the mazes could be chosen so that the common point of the three rods could trace a simple closed curve. Observe that none of the mazes may contain any cycles, or some pieces of the cube would fall out of the puzzle. So, what Lenstra was really asking for is a simple closed curve whose projections onto three pairwise orthogonal planes contain no cycles. In other words, he wanted the three shadows of a simple closed 3D curve to all be trees.

As Peter Winkler reported in his book *Mathematical mind-benders* [4], a solution had already been found some years before by John R. Rickard, who discovered the curve illustrated in Figure 2, also appearing on the front cover of Winkler's book.

Figure 1: Oskar's maze, produced by Bits and Pieces.

Several other, more complex solutions to Lenstra's problem are known. Notably, in 2012 Adam P. Goucher constructed a simple closed curve having shadows that are all trees, which also happens to be a trefoil knot [3]. The curve was therefore named *Treefoil*. Goucher also constructed a pair of linked cycles whose union has shadows that are all trees.

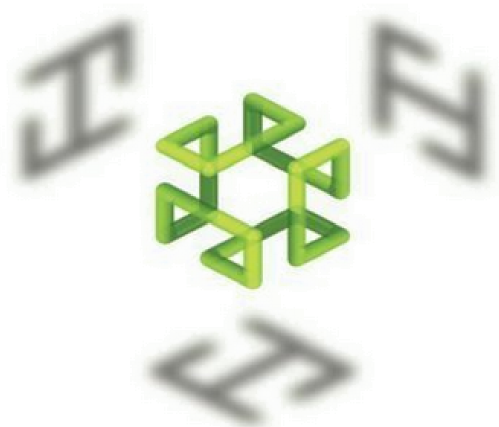

Figure 2: Rickard's curve, illustrated by Afra Zomorodian, and appearing on the front cover of Peter Winkler's book *Mathematical mind-benders*.

Our research is motivated by the following two questions. Is it possible for the three shadows of a simple closed curve to be paths, i.e., have neither cycles nor branch points? Can the three shadows of a simple open curve be simple closed curves? Both these questions are related to Lenstra's question, whose history is outlined

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in [4]. These questions have also been posed independently (see [1, 2]).

Our contribution. In Section 2 we answer the first question in the negative: the three shadows of a simple closed curve in $\mathbb{R}^3$ cannot all be paths.

In Section 3 we answer the second question in the affirmative: there exist simple open curves in $\mathbb{R}^3$ whose three shadows are simple closed curves, although the shadows cannot all be convex. Furthermore, we exhibit a polygonal chain with this property having only six vertices, and we prove that six is the minimum.

In Section 4 we extend Rickard's curve to higher dimensions, giving an inductive construction of a d-sphere embedded in $\mathbb{R}^{d+2}$, for every $d \geq 1$, whose $d+2$ shadows are all contractible. (A contractible set is one that can be continuously shrunk to a point, and hence it has no holes.)

Section 5 concludes the paper with some remarks and suggestions for further work.

This research has obvious applications in computer vision and 3D object reconstruction, where the goal is to deduce properties of an unknown 3-dimensional object given its three projections. Specifically, we may want to study the topology of an object that projects to three given paths. It is easy to see that such an object may not be unique, and hence it makes sense to study the set of 3-dimensional objects that are *compatible* with three given projections. Observe that any such set is closed under taking unions, and therefore it has a unique "largest" object, which is the union of all the objects in the set.

It is interesting to note that there are triplets of paths that are not compatible with any connected set, such as the one in Figure 3. This means that an Oskar's-maze-like puzzle could be "unsolvable" even if it had no crossroads on any face. By "unsolvable" we mean that the set of locations that are reachable by the central point of the three rods depends on where the rods are located. Therefore, if we assign two points in the 3-dimensional maze determined by the three 2-dimensional mazes, it may be impossible to go from one to the other by moving the rods around.

2 The shadows of a cycle

In this section we prove that the shadows of a simple closed curve in $\mathbb{R}^3$ cannot all be simple open curves. We start with some notation and definitions.

For a point $p \in \mathbb{R}^n$ and $1 \leq i \leq n$, we denote by p_i the i-th coordinate of p. The *x_i-projection*, or *x_i-shadow*, of a set $A \subseteq \mathbb{R}^n$, denoted by $\pi_i(A)$, is the orthogonal projection of A into the i-th coordinate hyperplane, e.g., $\pi_1(A) = \{(p_2, p_3, \cdots, p_n) \mid p \in A\}$. If $A = \{p\}$, we may simply write $\pi_i(p)$ instead of $\pi_i(\{p\})$.

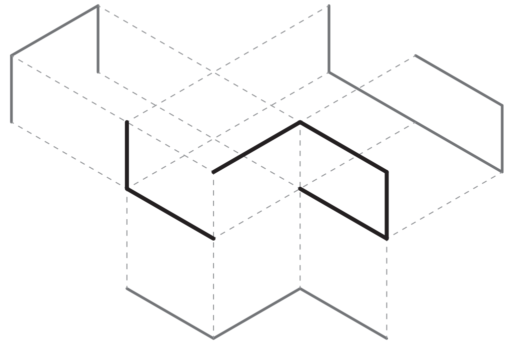

Figure 3: Connected shadows whose unique compatible set is disconnected.

A *path* is a (non-degenerate) simple open curve, and the *interior* γ° of a path γ is a copy of the path with its endpoints removed. A *cycle* is a (non-degenerate) simple closed curve.

An *x_i-strand* of a simple curve is a minimal path between the x_i-extremes of the curve. That is, an x_i-strand of a simple curve γ with x_i-minimum $a_i = \min_{x \in \gamma} x_i$ and x_i-maximum $b_i = \max_{x \in \gamma} x_i$ is a path $\sigma \subseteq \gamma$ whose endpoints s and t are such that $s_i = a_i$ and $t_i = b_i$, and every internal point $x \in \sigma^\circ$ is such that $x_i \neq a_i, b_i$.

Observation 1 *The interiors of any two distinct x_i-strands of a simple curve are disjoint. Hence any two distinct x_i-strands of a path intersect at most at one common endpoint.*

Observation 2 *If σ is an x_i-strand of a simple curve γ, then $\pi_j(\sigma)$ is an x_i-strand of $\pi_j(\gamma)$, for $j \neq i$.*

If $\pi_j(\gamma)$ is a path, the converse of Observation 2 is also true, as stated in the next lemma.

Lemma 1 *If σ is an x_i-strand of the x_j-projection of a simple curve γ, with $i \neq j$, and $\pi_j(\gamma)$ is a path, then there exists an x_i-strand of γ whose x_j-projection is σ.*

Proof. Let a and b be the endpoints of $\pi_j(\gamma)$, let $a', b' \in \gamma$ such that $\pi_j(a') = a$ and $\pi_j(b') = b$, and let $\gamma' \subseteq \gamma$ be a path with endpoints a' and b'. Since $\pi_j(\gamma)$ is a path, $\pi_j(\gamma) = \pi_j(\gamma')$. Let c and d be the endpoints of σ, such that a and c belong to the same connected component of $\pi_j(\gamma') \setminus \sigma^\circ$. Parameterizing γ' from a' to b', let c' be the last point of γ' such that $\pi_j(c') = c$. Because d separates c and b in $\pi_j(\gamma')$, there are points of γ' after c' whose x_j-projection is d. Letting d' be the first of such points, the sub-path of γ' with endpoints c' and d' is an x_i-strand of γ whose x_j-projection is σ. $\square$

In the following lemma we show that, if two shadows of a non-degenerate cycle are paths, then each of the two shadows has at least two similarly-oriented strands.

Lemma 2 *If γ is a cycle in $\mathbb{R}^3$ that is not contained in any x_1-orthogonal plane, and $\pi_2(\gamma)$ and $\pi_3(\gamma)$ are paths, then $\pi_3(\gamma)$ has at least two distinct x_1-strands.*

Proof. Since γ is not in an x_1-orthogonal plane, γ and $\pi_3(\gamma)$ both have at least one x_1-strand. Let σ be an x_1-strand of γ, let $\tau_2 = \pi_2(\sigma)$, and assume for contradiction that $\pi_3(\gamma)$ has a unique x_1-strand τ_3, as sketched in Figure 4.

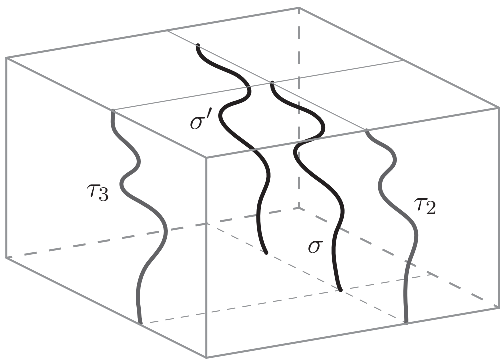

Figure 4: Some x_1-strands of the curve in Lemma 2.

Since τ_2 is an x_1-strand of $\pi_2(\gamma)$ by Observation 2, and the endpoints of a path cannot be in the interior of one of its strands, $\pi_2(\gamma) \setminus \tau_2^\circ$ contains the endpoints of $\pi_2(\gamma)$. Also, the x_2-shadow of $\gamma \setminus \sigma^\circ$ is a superset of $\pi_2(\gamma) \setminus \tau_2^\circ$, and hence it contains the endpoints of $\pi_2(\gamma)$, as well. Moreover, $\gamma \setminus \sigma^\circ$ is connected (because γ is a cycle), hence $\pi_2(\gamma \setminus \sigma^\circ)$ is a connected subset of the path $\pi_2(\gamma)$ containing its endpoints, and therefore it must be all of $\pi_2(\gamma)$. By Lemma 1, γ has an x_1-strand $\sigma' \in \gamma \setminus \sigma^\circ$ such that $\pi_2(\sigma') = \pi_2(\sigma) = \tau_2$. Since $\pi_3(\gamma)$ has a unique x_1-strand τ_3, we have $\pi_3(\sigma') = \pi_3(\sigma) = \tau_3$, again by Observation 2.

Respectively parameterize σ, σ', τ_2, and τ_3 each from the x_1-minimum a_1 to the x_1-maximum b_1 of γ. Choose some value c_1 strictly between these extremes, $a_1 < c_1 < b_1$. Let $s \in \sigma$, $s' \in \sigma'$, $t_2 \in \tau_2$, $t_3 \in \tau_3$ respectively be the first point of each strand where the x_1 coordinate attains the value c_1. With this we have $\pi_2(s) = \pi_2(s') = t_2$ and $\pi_3(s) = \pi_3(s') = t_3$, which implies $s = s'$. So the interiors of the strands σ and σ' intersect, contradicting Observation 1. Thus our assumption must be wrong: $\pi_3(\gamma)$ must have at least two distinct x_1-strands. □

Next we prove that an x_1-strand and an x_2-strand of a planar path must intersect each other, and therefore their union must be a sub-path.

Lemma 3 *If σ_1 and σ_2 are respectively an x_1-strand and an x_2-strand of a path γ in $\mathbb{R}^2$, then $\sigma_1 \cup \sigma_2$ is a path.*

Proof. Let $B = [a_1, b_1] \times [a_2, b_2]$ be the bounding box of γ, and let s_1 and t_1 be the leftmost and rightmost points of σ_1, respectively. Consider the polygonal chain τ with vertices s_1, $(a_1 - 1, a_2 - 1)$, $(b_1 + 1, a_2 - 1)$, t_1, in this order. Then $\sigma_1 \cup \tau$ is a cycle which, by the Jordan Curve Theorem, disconnects the plane into two components: an interior I and an exterior E.

Let s_2 and t_2 be the lowest and highest points of σ_2, respectively. Note that s_2 lies on the bottom edge of B, and hence it lies either in I or on the curve $\sigma_1 \cup \tau$. Similarly, t_2 lies on the top edge of B, and hence it lies either in E or on the curve $\sigma_1 \cup \tau$. Thus, $s_2 \notin E$ and $t_2 \notin I$. It follows that σ_2 must intersect $\mathbb{R}^2 \setminus (I \cup E) = \sigma_1 \cup \tau$. Since $\sigma_2 \subset B$ and $\tau^\circ \cap B = \varnothing$, σ_2 must intersect σ_1.

Thus, $\sigma_1 \cup \sigma_2$ is a connected subset of the path γ, and is therefore a path. □

In our final lemma we show that a planar path cannot have two distinct x_1-strands and two distinct x_2-strands.

Lemma 4 *A path in $\mathbb{R}^2$ has either a unique x_1-strand or a unique x_2-strand.*

Proof. Assume for a contradiction that γ is a path in $\mathbb{R}^2$ with distinct x_1-strands σ_1, σ_2 and distinct x_2-strands τ_1, τ_2. By Observation 1, σ_1 and σ_2 are either disjoint, or their intersection is precisely a common endpoint. Suppose for a contradiction that they are disjoint, and let $\sigma' \subset \gamma$ be the minimal path connecting them. By Lemma 3, $\sigma_1 \cup \tau_1$ is a path, as well as $\sigma_2 \cup \tau_1$, which implies that $\sigma' \subseteq \tau_1$. Similarly, $\sigma' \subseteq \tau_2$, and therefore $\sigma' \subseteq \tau_1 \cap \tau_2$, contradicting Observation 1. Thus $\sigma_1 \cap \sigma_2$ is a single point p, and by a symmetric argument $\tau_1 \cap \tau_2 = p$, as well. Let $B = [a_1, b_1] \times [a_2, b_2]$ be the bounding box of γ. Then p must be a vertex of B, and we may assume that $p = (b_1, b_2)$. Also, by symmetry, we may assume that $\tau_1 \subseteq \sigma_1$. It follows that $\sigma_1 \cap \tau_2 = \sigma_2 \cap \tau_1 = p$, and either $\tau_2 \subseteq \sigma_2$ or $\sigma_2 \subseteq \tau_2$.

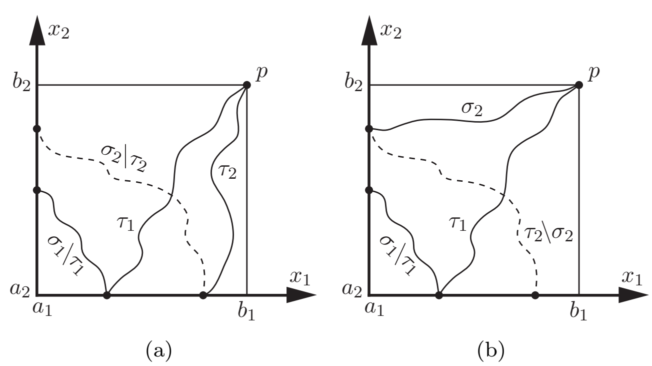

Figure 5: Cases of Lemma 4.

Suppose that $\tau_2 \subseteq \sigma_2$, as in Figure 5(a). Then τ_2° is in the same connected component of $B \setminus \tau_1$ as the edge $\{b_1\} \times [a_2, b_2)$. This implies that $\sigma_2 \setminus \tau_2$ intersects τ_1, contradicting the fact that $\sigma_2 \cap \tau_1 = p$.

Suppose that $\sigma_2 \subseteq \tau_2$, as in Figure 5(b). Then σ_2° is in the same connected component of $B \setminus \sigma_1$ as the edge $[a_1, b_1) \times \{b_2\}$. This implies that $\tau_2 \setminus \sigma_2$ intersects σ_1, contradicting the fact that $\sigma_1 \cap \tau_2 = p$.

Thus our assumption fails: γ has either a unique x_1-strand or a unique x_2-strand. □

We are now able to prove the main result of this section.

Theorem 5 *There is no cycle in $\mathbb{R}^3$ whose shadows are all paths.*

Proof. Assume for a contradiction that the three shadows of a cycle γ in $\mathbb{R}^3$ are all paths. Note that γ cannot lie in any x_i-orthogonal plane, or $\pi_i(\gamma)$ would not be a path. By Lemma 2, since the x_2-shadow and the x_3-shadow are both paths, the x_3-shadow must have at least two distinct x_1-strands. Likewise, since the x_1-shadow and the x_3-shadow are both paths, the x_3-shadow must also have at least two distinct x_2-strands. But by Lemma 4, a path in the (x_1, x_2)-plane cannot have two distinct x_1-strands and two distinct x_2-strands, which is a contradiction. □

3 The shadows of a path

Here we study the simple open curves in $\mathbb{R}^3$ whose shadows are simple closed curves. In contrast with the similarly-defined curves of the previous section, in this case we can construct a wealth of such curves. An example is illustrated in Figure 6.

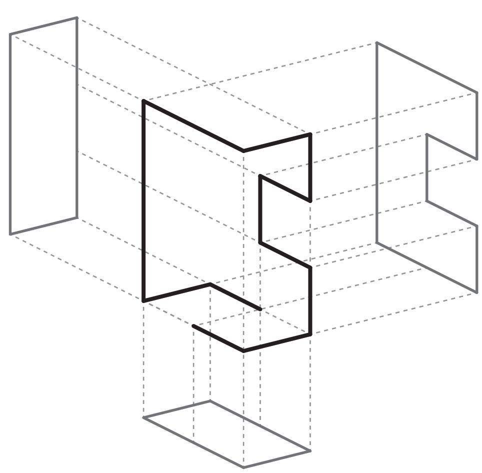

Figure 6: Axis-aligned polygonal path whose shadows are all cycles.

Note that the curve in Figure 6 is a *polygonal path* (i.e., a simple open polygonal chain) consisting of axis-parallel segments. If we allow arbitrarily oriented segments, we can find an example with only six vertices, which is the minimum possible.

Theorem 6 *There exists a polygonal path in $\mathbb{R}^3$ with six vertices whose shadows are cycles. No such polygonal path exists with fewer than six vertices.*

Proof. An example of such a polygonal path is $(1,0,1)\,(0,0,0)\,(1,1,0)\,(0,3,0)\,(2,0,2)\,(1,0,0)$, which is shown in Figure 7.

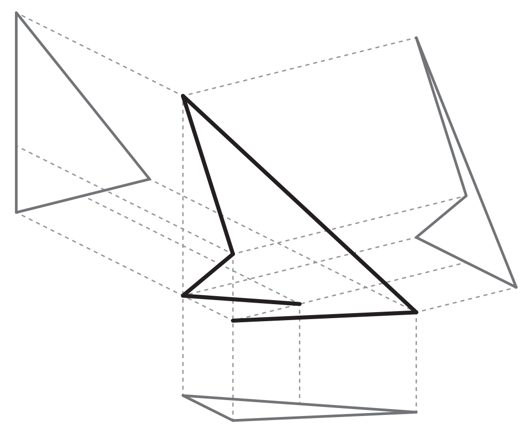

Figure 7: Minimal polygonal path whose shadows are all cycles.

Suppose for a contradiction that a polygonal path in $\mathbb{R}^3$ with $n < 6$ vertices exists such that its shadows are cycles. If $n \leq 3$, then clearly no shadow can be a cycle. Suppose that $n = 4$, and let the polygonal path be $v_1\, v_2\, v_3\, v_4$. Then each shadow must be a triangle, and hence $\pi_i(v_1) = \pi_i(v_4)$ for every $i \in \{1, 2, 3\}$. It follows that $v_1 = v_4$, which contradicts the fact that a polygonal path is an open curve.

Assume now that $n = 5$, and the polygonal path is $v_1\, v_2\, v_3\, v_4\, v_5$. For every $i \in \{1, 2, 3\}$, the x_1-projection of the polygonal path is either a triangle or a quadrilateral. In both cases, the x_i-shadows of the segments $v_1\, v_2$ and $v_4\, v_5$ have a non-empty intersection. Since $v_1 \neq v_5$, the x_i-shadows of v_1 and v_5 do not coincide for at least two i's, say, $i = 1$ and $i = 2$. Then the x_1-shadow of the polygonal path must be a triangle, $\pi_i(v_1\, v_2)$ and $\pi_i(v_4\, v_5)$ are collinear, and hence the segments $v_1\, v_2$ and $v_4\, v_5$ lie on a plane that is orthogonal to the (x_2, x_3)-plane. Similarly, the segments $v_1\, v_2$ and $v_4\, v_5$ lie on a plane that is orthogonal to the (x_1, x_3)-plane, too. Hence $v_1\, v_2$ and $v_4\, v_5$ are either collinear or they lie on a common x_3-orthogonal plane. If $v_1\, v_2$ and $v_4\, v_5$ are collinear (and disjoint), then their x_i-shadows are disjoint for some $i \in \{1, 2, 3\}$, contradicting the fact that their intersection must be non-empty. If $v_1\, v_2$ and $v_4\, v_5$ lie on a common x_3-orthogonal plane, then $\pi_3(v_1\, v_2)$ and $\pi_3(v_4\, v_5)$ are disjoint, which is again a contradiction. □

Note that, in all the above examples, one of the shadows is a non-convex cycle. It is natural to ask whether a path exists whose shadows are all convex cycles. In the following theorem, we answer in the negative. (Due to space constraints, we only give a sketch of the proof.)

Theorem 7 *There is no path in $\mathbb{R}^3$ whose shadows are convex cycles.*

Proof (sketch). Suppose for contradiction that there exists a path γ in $\mathbb{R}^3$ whose shadows are convex cycles. For every $i \in \{1,2,3\}$, γ lies on the surface Γ_i of a cylinder with section $\pi_i(\gamma)$ and x_i-parallel axis.

The intersection of Γ_1 and Γ_2 is sketched in Figure 8. It consists of two horizontal axis-aligned rectangles R_1 and R_2 (assuming that the vertical direction is x_3-parallel) whose vertices are joined by four paths σ_1, σ_2, σ_3, and σ_4. The rectangles R_1 and R_2 may be degenerate, i.e., they may be x_1-parallel or x_2-parallel segments, or points. Let a horizontal plane intersect the interior of the path σ_i in the point s_i, for each $i \in \{1,2,3,4\}$. Then, for all $i \in \{1,2,3,4\}$, either $s_i \in \gamma$ or $s_{i+1} \in \gamma$, where indices are taken modulo 4.

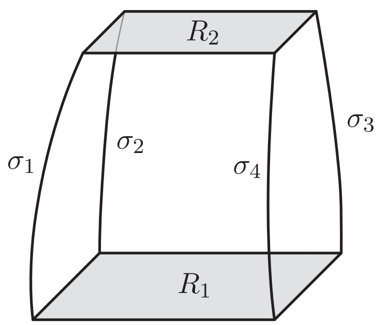

Figure 8: Intersection of Γ_1 and Γ_2.

Further intersecting $\Gamma_1 \cap \Gamma_2$ with Γ_3, we reduce R_1 and R_2 to at most four horizontal curves each. Therefore, in total we have $n \leq 12$ curves, whose union is an embedding in $\mathbb{R}^3$ of a graph G with n edges. Also, we may assume without loss of generality that each endpoint of γ lies at a vertex of the embedding of G, or at the midpoint of one of the n edges. Hence there are only finitely many possible graphs G to consider, and only finitely many choices of γ in each graph embedding. By exhaustively examining all the possible choices of γ, we conclude that none of them has shadows that are all convex cycles. □

4 Shadows in higher dimensions

In this section we generalize Rickard's curve to higher dimensions. We inductively construct an embedding of a d-sphere in $\mathbb{R}^{d+2}$ whose $d+2$ shadows are all contractible, i.e., they deformation-retract to a point.

An x_i-*slice* of a set $A \subseteq \mathbb{R}^n$, with $1 \leq i \leq n$, is a non-empty intersection between A and an x_i-orthogonal hyperplane.

Theorem 8 *For every $d \geq 1$, there exists an embedding of a d-sphere in $\mathbb{R}^{d+2}$ whose shadows are all contractible.*

Proof. Let S_1 be Rickard's curve, introduced in Section 1. Then, for all $d \geq 1$, we inductively define

$$S_{d+1} = \bigcup_{\lambda \in [-1,1]} (1-|\lambda|) \cdot S_d \times \{\lambda\}.$$

It is easy to see that S_d is an embedding of a d-sphere in $\mathbb{R}^{d+2}$ for every $d \geq 1$. We claim that all the shadows of S_d deformation-retract to the point $\{0\}^{d+1}$. This is true for $d = 1$, as suggested by Figure 2. Assume now the inductive hypothesis that the claim is true for S_d, and therefore there exists a continuous map

$$F_{d,i} \colon \pi_i(S_d) \times [0,1] \to \pi_i(S_d)$$

with $F_{d,i}(x,0) = x$ and $F_{d,i}(x,1) = \{0\}^{d+1}$, for every $1 \leq i \leq d+2$. Now, for each $1 \leq i \leq d+3$, we can construct a continuous map

$$F_{d+1,i} \colon \pi_i(S_{d+1}) \times [0,1] \to \pi_i(S_{d+1})$$

with $F_{d+1,i}(x,0) = x$ and $F_{d+1,i}(x,1) = \{0\}^{d+2}$.

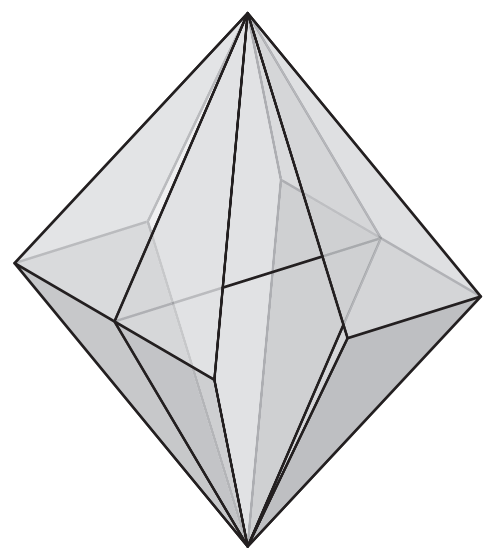

Figure 9: x_i-shadow of S_2, for $1 \leq i \leq 3$.

If $1 \leq i \leq d+2$, we first define the auxiliary map

$$F' \colon \pi_i(S_{d+1}) \times [0,1] \to \pi_i(S_{d+1})$$

as follows. For every $x \in \pi_i(S_{d+1})$ such that $|x_{d+2}| \neq 1$ and $\lambda \in [0,1]$, we let

$$F'(x,\lambda) = (1-|x_{d+2}|) \cdot F_{d,i}\left(\frac{\pi_{d+2}(x)}{1-|x_{d+2}|}, \lambda\right) \times \{x_{d+2}\}.$$

If $x \in \pi_i(S_{d+1})$ with $|x_{d+2}| = 1$ and $\lambda \in [0,1]$, we let $F'(x,\lambda) = x$. Observe that every x_{d+2}-slice of $\pi_i(S_{d+1})$

is a scaled copy of $\pi_i(S_d)$. (Figure 9 shows $\pi_i(S_{d+1})$ for $d = 1$.) Informally, F' applies $F_{d,i}$ with parameter λ to a suitably scaled copy of each x_{d+2}-slice, and then it rescales it back. Therefore, since $F_{d,i}$ is a deformation retraction of $\pi_i(S_d)$ to the point $\{0\}^{d+1}$, F' is a deformation retraction of $\pi_i(S_{d+1})$ to the segment $\{0\}^{d+1} \times [-1, 1]$. To obtain $F_{d+1,i}$, one just has to compose F' with a deformation retraction of $\{0\}^{d+1} \times [-1, 1]$ to the point $\{0\}^{d+2}$. In formulas, for $x \in \pi_i(S_{d+1})$ and $\lambda \in [0, 1]$,

$$F_{d+1,i}(x, \lambda) = \begin{cases} F'(x, 2\lambda) & \text{if } \lambda < 1/2 \\ (2 - 2\lambda) \cdot F'(x, 1) & \text{if } \lambda \geq 1/2. \end{cases}$$

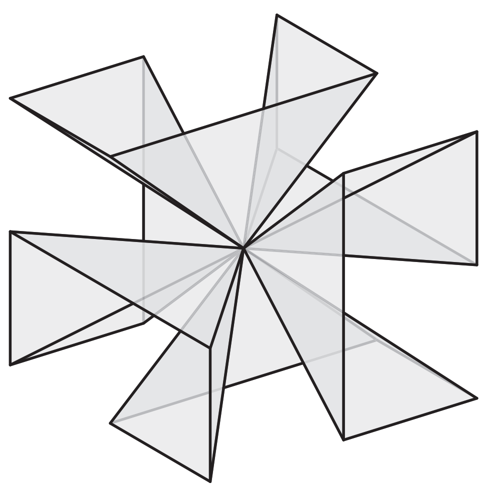

Figure 10: x_4-shadow of S_2.

If $i = d + 3$, we can simply set

$$F_{d+1,i}(x, \lambda) = (1 - \lambda) \cdot x$$

for every $x \in \pi_i(S_{d+1})$ and $\lambda \in [-1, 1]$. This is easily seen to be a deformation retraction to $\{0\}^{d+2}$. (Figure 10 shows $\pi_i(S_{d+1})$ for $d = 1$.)

Hence all the shadows of the d-sphere S_d deformation-retract to a point for every $d \geq 1$, meaning that they are contractible. □

5 Concluding remarks

In this paper we studied the shadows of curves in $\mathbb{R}^3$ (a shadow being an axis-parallel projection), also settling some long-standing open problems posed in [1, 2].

In Section 2 we proved that there is no cycle in $\mathbb{R}^3$ whose shadows are all paths. Note that by applying a projective transformation, we may equivalently define shadows to be perspective projections, provided that the three viewpoints are not collinear, and the plane through them does not intersect the curve.

In Section 3 we proved that there exist paths in $\mathbb{R}^3$ whose shadows are all cycles. We also showed that, if such a path is a polygonal chain, it must have at least six vertices, and we found an example with exactly six vertices. Then we proved that there is no path in $\mathbb{R}^3$ whose shadows are all convex cycles.

Finally, in Section 4 we showed that there exists an embedding of a d-sphere in $\mathbb{R}^{d+2}$ whose shadows are all contractible, for every $d \geq 1$. This generalizes Rickard's curve (see Figure 2), which is a cycle in $\mathbb{R}^3$ whose shadows contain no cycles.

Our results can be expanded in several directions. A natural goal would be to minimize the total number of branch points of the shadows of a cycle in $\mathbb{R}^3$, assuming that all shadows are cycle-free. Because each shadow of Rickard's curve has two branch points, such a minimum is at most six. On the other hand, by Theorem 5, the minimum is at least one. With the same proof technique employed in Section 2, we can prove the following generalized version of Theorem 5, which implies that the minimum number of branch points of the shadows must be at least three.

Theorem 9 *There is no cycle γ in $\mathbb{R}^3$ with cycle-free shadows such that $\pi_1(\gamma)$ is a path, and $\pi_2(\gamma)$ and $\pi_3(\gamma)$ have at most one branch point each.* □

We conjecture Rickard's curve to be an optimal example in terms of branch points of its shadows.

Conjecture 1 *If the shadows of a cycle in $\mathbb{R}^3$ are all cycle-free, then each shadow has at least two branch points.*

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