Inscribing $\mathcal{H}$-Polyhedra in Quadrics Using a Projective Generalization of Closed Sets*

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Abstract

We present a projective generalization of closed sets, called *complete projective embeddings*, which allows us to inscribe $\mathcal{H}$-polyhedra in quadrics efficiently. Essentially, the complete projective embedding of a closed convex set $\mathbf{P} \subseteq \mathbb{K}^d$ is a double cone in $\mathbb{K}^{d+1}$. We show that complete projective embeddings of polyhedral sets are of particular interest and already occurred in the theory of linear fractional programming. Our approach works as follows: By projective principal axis transformation the quadric is converted to a hyperboloid and then approximated by an inner (right) spherical cylinder. Now, given an inscribed $\mathcal{H}$-polytope of the spherical cross section, cylindrification of the polyhedron yields an inscribed $\mathcal{H}$-polyhedron of the spherical cylinder and, hence, of the hyperboloid. After application of the inverse base transformation this approach finally yields an inscribed set of the quadric. The crucial task of this procedure is to find an appropriate generalization of closed sets, which is closed under the involved projective transformations and allows us to recover the non-projective equivalents to the inscribed sets obtained by our approach. It turns out that complete projective embeddings are the requested generalizations.

1 Introduction

During our research for an efficient method to generate inscribed $\mathcal{H}$-polyhedra of quadrics, we developed the notion of *complete projective embeddings* of closed sets. A complete projective embedding of a closed convex set $\mathbf{P} \subseteq \mathbb{K}^d$ is essentially a double cone in $\mathbb{K}^{d+1}$ that is obtained by translating $\mathbf{P}$ onto the hyperplane $\{\binom{\mathbf{x}}{\lambda} \mid \lambda = 1\} \subseteq \mathbb{K}^{d+1}$ and consists of all lines joining the origin and some point of the translated set. Certainly, we can always recover the original set by restricting the double cone to its intersection with the designated hyperplane $\{\binom{\mathbf{x}}{\lambda} \mid \lambda = 1\}$. As we shall apply linear maps on the double cones, we are interested in what happens to the intersections of the images with the designated hyperplane – a question which is very similar to the classical theory of conic sections. For the important case where $\mathbf{P}$ is an $\mathcal{H}$-polyhedron we obtain that each of this intersections can be described by a union of two polyhedra having representation matrices of certain symmetry.

In 2009 Gallier described the double cone representation of polyhedra which he calls *projective polyhedra* [5]. It turns out that projective polyhedra are complete projective embedding of polyhedral sets, and for these we shall use both notions synonymously. In his introduction he noted that "*to the best of our knowledge, this notion of projective polyhedron is new*". Although we neither found any systematic work on this notion, we discovered that projective polyhedra are closely related to the sets of feasible solutions of linear fractional programs as given by Charnes and Cooper [2] in 1962. This paper is a contribution to show the theoretical and practicable relevance of projective polyhedra and their generalization – the complete projective embeddings of closed sets – and aims for establishing them as independent objects of investigation.

In the second part of the paper we show how complete projective embeddings can be used to generate inscribed $\mathcal{H}$-polyhedra in quadrics. The research is motivated by a combination of reachability analysis and stability analysis for linear systems[1]: In reachability analysis of linear systems one typically computes tight overapproximations of the reachable states until all trajectories either leave the region of admissible states or a trajectory enters a certain set **Avoid**. The computation is based on a convex representation of flow segments over a time interval of short length. The convex set representation is highly compatible with $\mathcal{H}$-polyhedra. This holds for approaches using support functions as well as for our particular approach where we have chosen *symbolic orthogonal projections* as a representation of polyhedral sets. Symbolic orthogonal projections allow an exact and efficient representation and evaluation of typical geometric operations occurring in reachability analysis, including Minkowski sums, convex hulls, intersections, and arbitrary affine transformations [7].

On the other hand, stability analysis provides Lyapunov functions in terms of quadratic forms. For each

*This work has been partly supported by the German Research Foundation (DFG) as part of the Transregional Collaborative Research Center "Automatic Verification and Analysis of Complex Systems" (SFB/TR 14 AVACS, `www.avacs.org`).

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[1] Actually we deal with hybrid systems. However, for an explanation of our motivation it suffices to restrict to linear systems.

admissible state of the system the value of this quadratic form is positive and decreases monotonically along each trajectory where the value of the equilibrium point is 0. A level set of a quadratic form consists of all states whose value is below a certain level. Hence, its boundary is given by a quadric. Assume, there exists a lower bound b on the value of the quadratic form for all states in **Avoid**. If the current flow segment is completely located within a level set of level b, we conclude that there will no further intersection with **Avoid**, and we may stop the reachability analysis. Alas, while there exists efficient methods to check whether the current flow segment – either represented by support functions or symbolic orthogonal projections – is contained in an $\mathcal{H}$-polyhedron, we have no efficient method to test whether the current flow segment is contained in a level set. Hence, we use an inner approximation of the level set in terms of an $\mathcal{H}$-polyhedron to assess the inclusion of the current flow segment instead. For further details we point the interested reader to [8].

2 Preliminaries

By $\mathbb{K}$ we denote an arbitrary ordered field. The corresponding euclidean field is denoted by $\mathbb{E}$. A closed convex polyhedron $\mathbf{P}$ is the set $\mathbf{P} = \mathbf{P}(A, \mathbf{a}) = \{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{a}\}$. In the following the term *polyhedron* will always refer to a closed convex polyhedron. Vectors and sets of vectors are denoted by bold letters. All coefficients of the vectors $\mathbf{0}$ and $\mathbf{1}$ are 0 or 1, respectively. We use the superscript ^{T} to indicate the transpose of a vector or matrix. The notion A^{-T} denotes the transpose of the inverse of the matrix A. The coefficients of a d-dimensional vector $\mathbf{x}$ are denoted by $x_1, \ldots, x_d$.

2.1 Projective Space

We give a short introduction into projective geometry as it can be found in many textbooks like [6, 1]. The *projective space* $\mathrm{Proj}^d(\mathbb{K})$ *induced by* $\mathbb{K}$ can be identified with the set $\mathbb{K}^{d+1} \setminus \{\mathbf{0}\}$ where two vectors $\mathbf{x}$, $\mathbf{y}$ are *projectively equal* if and only if there exists some $\lambda \neq 0$ such that $\mathbf{x} = \lambda\mathbf{y}$, formally

$$\mathbf{x} =_p \mathbf{y} \iff \exists\lambda \neq 0\colon \mathbf{x} = \lambda\mathbf{y}. \tag{1}$$

The injective mapping $\iota\colon \mathbb{K}^d \to \mathrm{Proj}^d(\mathbb{K})$, $\mathbf{x} \mapsto \binom{\mathbf{x}}{1}$ defines the *canonical embedding* of $\mathbb{K}^d$ into the projective space $\mathrm{Proj}^d(\mathbb{K})$. On the other hand, any point $\binom{\mathbf{x}}{\lambda} \in \mathrm{Proj}^d(\mathbb{K})$ either represents the point $\frac{1}{\lambda}\mathbf{x}$ in $\mathbb{K}^d$ if and only if $\lambda \neq 0$, or $\binom{\mathbf{x}}{\lambda}$ represents a point at infinity, which is a point that has no corresponding point in $\mathbb{K}^d$. Actually, $\binom{\mathbf{x}}{0}$ represents $\lim_{\lambda\to 0^+} \frac{1}{\lambda}\mathbf{x}$, which coincides with $\lim_{\lambda\to 0^-} \frac{1}{\lambda}\mathbf{x}$ by projective equality.

3 Complete Projective Embeddings

The classical approach allows us to treat projective sets as subsets of $\mathbb{K}^{d+1} \setminus \{\mathbf{0}\}$, but it has a drawback: Due to the existential quantifier in (1) it is laborious to assess projective subset relations in $\mathbb{K}^{d+1} \setminus \{\mathbf{0}\}$. We shall introduce the notion of *projective completeness*, which allows us to assess projective subset relations in a simple way. The projective vector space $\mathrm{Proj}^d(\mathbb{K})$ is canonically embedded in the vector space $\mathbb{K}^{d+1} \setminus \{\mathbf{0}\}$ and, hence, also embedded in $\mathbb{K}^{d+1}$.

Definition 1 *Let $\mathbf{P}$ be a subset of $\mathbb{K}^{d+1}$. If (i) $\mathbf{0} \in \mathbf{P}$ and (ii) $\mathbf{x} \in \mathbf{P}$ implies that $\lambda\mathbf{x} \in \mathbf{P}$ for all $\lambda \in \mathbb{K}$, then we call $\mathbf{P}$* projectively complete.

Definition 2 *Let $\mathbf{P}$ be a closed set, and let $\tilde{\mathbf{P}}$ be the least closed and projectively complete set with $\iota(\mathbf{P}) \subseteq \tilde{\mathbf{P}}$. The set $\tilde{\mathbf{P}}$ is called the* complete projective embedding *of $\mathbf{P}$.*

Hence, the projectively complete embedding $\tilde{\mathbf{P}}$ of $\mathbf{P}$ always contains $\mathbf{0}$ and all projective representations $\binom{\mathbf{x}}{\lambda}$, $\lambda \neq 0$, of a point $\mathbf{x} \in \mathbf{P}$. Furthermore, since the union of two closed sets $\mathbf{P}_1$ and $\mathbf{P}_2$ is closed again, the complete projective embedding of $\mathbf{P}_1 \cup \mathbf{P}_2$ is the union $\tilde{\mathbf{P}}_1 \cup \tilde{\mathbf{P}}_2$ of the complete projective embeddings $\tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_2$.

Proposition 3 *Let $\mathbf{P}$, $\mathbf{Q}$ be two closed sets and $\tilde{\mathbf{P}}$, $\tilde{\mathbf{Q}}$ their complete projective embeddings. Then $\mathbf{P} \subseteq \mathbf{Q}$ if and only if $\tilde{\mathbf{P}} \subseteq \tilde{\mathbf{Q}}$.*

Proof. Assume $\mathbf{P} \subseteq \mathbf{Q}$ holds. Let $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$. In the case $\lambda \neq 0$ we have $\frac{1}{\lambda}\mathbf{x} \in \mathbf{P}$. Since $\mathbf{P} \subseteq \mathbf{Q}$, we also have $\frac{1}{\lambda}\mathbf{x} \in \mathbf{Q}$. The set $\tilde{\mathbf{Q}}$ is the complete projective embedding of $\mathbf{Q}$. Hence, $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{Q}}$. In the case $\lambda = 0$ we either have $\mathbf{x} = \mathbf{0}$ and $\binom{\mathbf{x}}{\lambda} = \binom{\mathbf{0}}{0}$ is trivially contained in $\tilde{\mathbf{Q}}$, or there exists a sequence $\left(\binom{\mathbf{x}_i}{\lambda_i}\right)_{i\in\mathbb{N}} \subseteq \tilde{\mathbf{P}}$ with $\lim_{i\to\infty} \binom{\mathbf{x}_i}{\lambda_i} = \binom{\mathbf{x}}{0}$ and $\lambda_i \neq 0$ for all $i \in \mathbb{N}$ since $\tilde{\mathbf{P}}$ is the least closed and projective complete set containing $\iota(\mathbf{P})$. We already have seen that any $\binom{\mathbf{x}_i}{\lambda_i} \in \tilde{\mathbf{P}}$ with $\lambda_i \neq 0$ is also in $\tilde{\mathbf{Q}}$. Furthermore, $\tilde{\mathbf{Q}}$ is closed. Hence, $\binom{\mathbf{x}}{0} \in \tilde{\mathbf{Q}}$. We have shown that $\mathbf{P} \subseteq \mathbf{Q}$ implies $\tilde{\mathbf{P}} \subseteq \tilde{\mathbf{Q}}$.

On the other hand, assume $\tilde{\mathbf{P}} \subseteq \tilde{\mathbf{Q}}$. Let $\mathbf{x} \in \mathbf{P}$. Then $\binom{\mathbf{x}}{1} \in \tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}} \subseteq \tilde{\mathbf{Q}}$ implies $\binom{\mathbf{x}}{1} \in \tilde{\mathbf{Q}}$. Hence, $\mathbf{x} \in \mathbf{Q}$. We have shown that $\tilde{\mathbf{P}} \subseteq \tilde{\mathbf{Q}}$ implies $\mathbf{P} \subseteq \mathbf{Q}$. $\square$

Proposition 4 *Let $\mathbf{P} = \mathbf{P}(A, \mathbf{a}) = \left\{\mathbf{x} \in \mathbb{K}^d \,\middle|\, A\mathbf{x} \leq \mathbf{a}\right\}$ be a polyhedron, and let $\tilde{\mathbf{P}} = \mathbf{P}_1 \cup \mathbf{P}_2$, where*

$$\mathbf{P}_1 = \mathbf{P}\left(\begin{pmatrix} A & -\mathbf{a} \\ \mathbf{0}^T & -1 \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}\right),$$
$$\mathbf{P}_2 = \mathbf{P}\left(\begin{pmatrix} -A & \mathbf{a} \\ \mathbf{0}^T & 1 \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}\right).$$

Then $\tilde{\mathbf{P}}$ is the complete projective embedding of $\mathbf{P}$.

Proof. Obviously, we have $\iota(\mathbf{P}) \subseteq \mathbf{P}_1 \subseteq \tilde{\mathbf{P}}$.

We show that $\tilde{\mathbf{P}}$ is projectively complete. Obviously, $\binom{\mathbf{0}}{0} \in \tilde{\mathbf{P}}$. Let $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$, i.e., $A\mathbf{x} - \lambda\mathbf{a} \leq \mathbf{0}$, $-\lambda \leq 0$ or $-A\mathbf{x} + \lambda\mathbf{a} \leq \mathbf{0}$, $\lambda \leq 0$. Let $\mu \in \mathbb{K}$. Either we have $\mu \geq 0$ or $\mu < 0$. In both cases we obtain $A\mu\mathbf{x} - \mu\lambda\mathbf{a} \leq \mathbf{0}$, $-\mu\lambda \leq 0$ or $-A\mu\mathbf{x} + \mu\lambda\mathbf{a} \leq \mathbf{0}$, $\mu\lambda \leq 0$. Hence, $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$ implies $\mu\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$ for all $\mu \in \mathbb{K}$.

We show that $\tilde{\mathbf{P}}$ is a closed set. Let $\left(\binom{\mathbf{x}_i}{\lambda_i}\right)_{i\in\mathbb{N}}$ be a sequence with $\binom{\mathbf{x}_i}{\lambda_i} \in \tilde{\mathbf{P}}$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty}\binom{\mathbf{x}_i}{\lambda_i} = \binom{\mathbf{x}}{\lambda}$. We have to show that $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$. In the case $\lambda \neq 0$ let J be the set of all indices i where $\lambda_i \neq 0$. Then $(\frac{1}{\lambda_i}\mathbf{x}_i)_{i\in J}$ is an infinite sequence of members in $\mathbf{P}$ which converges to $\frac{1}{\lambda}\mathbf{x}$. The polyhedron $\mathbf{P}$ is closed, hence $\frac{1}{\lambda}\mathbf{x} \in \mathbf{P}$ and $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$. Let $\lambda = 0$. Either we have an infinite subset $J \subseteq \mathbb{N}$ such that $\lambda_i = 0$ for all $i \in J$. Then $A\mathbf{x}_i \leq \mathbf{0}$ or $-A\mathbf{x}_i \leq \mathbf{0}$ for all $i \in J$. The sets $\{\mathbf{y} \mid A\mathbf{y} \leq \mathbf{0}\}$ and $\{\mathbf{y} \mid -A\mathbf{y} \leq \mathbf{0}\}$ are closed, and so is their union. It follows $A\mathbf{x} \leq \mathbf{0}$ or $-A\mathbf{x} \leq \mathbf{0}$ and by definition of $\tilde{\mathbf{P}}$ also $\binom{\mathbf{x}}{0} \in \tilde{\mathbf{P}}$. Otherwise, there exists an index i_0 with $\lambda_i \neq 0$ for all $i \geq i_0$. The remaining sequence $\left(\binom{\mathbf{x}_i}{\lambda_i}\right)_{i\geq i_0}$ has the same limes. Hence, without loss of generality, we may assume $\lambda_i \neq 0$ for all $i \in \mathbb{N}$. Setting $\mathbf{y}_i = \mathbf{x}_i - \frac{\lambda_i}{\lambda_0}\mathbf{x}_0$ yields the equalities $\binom{\mathbf{y}_i}{0} = \binom{\mathbf{x}_i}{\lambda_i} - \frac{\lambda_i}{\lambda_0}\binom{\mathbf{x}_0}{\lambda_0}$ for all $i \in \mathbb{N}$. Since $\lim_{i\to\infty}\lambda_i = 0$ and $\lim_{i\to\infty}\mathbf{x}_i = \mathbf{x}$, we obtain $\lim_{i\to\infty}\frac{\lambda_i}{\lambda_0}\mathbf{x}_0 = \mathbf{0}$ and $\lim_{i\to\infty}\binom{\mathbf{y}_i}{0} = \lim_{i\to\infty}\binom{\mathbf{x}_i}{\lambda_i} - \lim_{i\to\infty}\frac{\lambda_i}{\lambda_0}\binom{\mathbf{x}_0}{\lambda_0} = \binom{\mathbf{x}}{0}$. Hence, $\binom{\mathbf{y}_i}{0}$ is a sequence with $\lim_{i\to\infty}\binom{\mathbf{y}_i}{0} = \binom{\mathbf{x}}{0}$ where the last coefficient of all members is equal to zero. For this case we already have shown that $\binom{\mathbf{x}}{0} \in \tilde{\mathbf{P}}$.

It remains to show that $\tilde{\mathbf{P}}$ is the least closed set which includes $\iota(\mathbf{P})$ and is projectively complete. Assume, $\tilde{\mathbf{P}}'$ is an arbitrary closed set that includes $\iota(\mathbf{P})$ and is projectively complete. We have to show that any $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$ is also contained in $\tilde{\mathbf{P}}'$. Therefore, let $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}$. In the case $\lambda \neq 0$ we have $\frac{1}{\lambda}\mathbf{x} \in \mathbf{P}$. Then $\frac{1}{\lambda}\mathbf{x} \in \iota(\mathbf{P})$ and, hence, $\frac{1}{\lambda}\mathbf{x} \in \tilde{\mathbf{P}}'$. Since $\tilde{\mathbf{P}}'$ is projectively complete, it follows $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{P}}'$. In the case $\lambda = 0$ we have $A\mathbf{x} \leq \mathbf{0}$ or $-A\mathbf{x} \leq \mathbf{0}$. Hence, for all $\mathbf{y} \in \mathbf{P}$, i.e., $A\mathbf{y} \leq \mathbf{a}$, we have $\mathbf{y} + \mu\mathbf{x} \in \mathbf{P}$ for all $\mu \geq 0$ or $\mathbf{y} + \mu\mathbf{x} \in \mathbf{P}$ for all $\mu \leq 0$. The set $\tilde{\mathbf{P}}'$ includes $\iota(\mathbf{P})$, hence, $\binom{\mathbf{y}+\mu\mathbf{x}}{1} \in \tilde{\mathbf{P}}'$ for all $\mu \geq 0$ or $\binom{\mathbf{y}+\mu\mathbf{x}}{1} \in \tilde{\mathbf{P}}'$ for all $\mu \leq 0$. In the former case let $\mu_i = i$ and in the latter case let $\mu_i = -i$. Since $\tilde{\mathbf{P}}'$ is projectively complete, we have $\frac{1}{\mu_i}\binom{\mathbf{y}+\mu_i\mathbf{x}}{1} \in \tilde{\mathbf{P}}'$ for all $i \geq 1$. Hence, $\lim_{i\to\infty}\frac{1}{\mu_i}\binom{\mathbf{y}+\mu_i\mathbf{x}}{1} = \lim_{i\to\infty}\binom{\frac{1}{\mu_i}\mathbf{y}+\mathbf{x}}{\frac{1}{\mu_i}} = \binom{\mathbf{x}}{0}$. Since $\tilde{\mathbf{P}}'$ is closed, it follows $\binom{\mathbf{x}}{0} \in \tilde{\mathbf{P}}'$. $\square$

Proposition 5 *Let $\tilde{\mathbf{P}}$ be the union of two cones*

$$\tilde{\mathbf{P}} = \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\end{pmatrix}, \mathbf{0}\right) \cup \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\end{pmatrix}, \mathbf{0}\right).$$

Then $\tilde{\mathbf{P}}$ is the complete projective embedding of $\mathbf{P}(A, \mathbf{a}) \cup \mathbf{P}(-A, -\mathbf{a})$.

Proof. Let $\tilde{\mathbf{R}}$ be the complete projective embedding of $\mathbf{P}(A, \mathbf{a}) \cup \mathbf{P}(-A, -\mathbf{a})$, i.e.,

$$\begin{aligned}\tilde{\mathbf{R}} = \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\\ \mathbf{0}^T & -1\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\0\end{pmatrix}\right) \cup \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\\ \mathbf{0}^T & 1\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\0\end{pmatrix}\right) \cup\\ \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\\ \mathbf{0}^T & -1\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\0\end{pmatrix}\right) \cup \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\\ \mathbf{0}^T & 1\end{pmatrix}, \begin{pmatrix}\mathbf{0}\\0\end{pmatrix}\right)\\ = \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\end{pmatrix}, \mathbf{0}\right) \cup \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\end{pmatrix}, \mathbf{0}\right).\end{aligned}$$

Hence, $\tilde{\mathbf{R}} = \tilde{\mathbf{P}}$. $\square$

Any complete projective embedding $\tilde{\mathbf{P}}$ of the form $\tilde{\mathbf{P}} = \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\end{pmatrix}, \mathbf{0}\right) \cup \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\end{pmatrix}, \mathbf{0}\right)$ is a double cone. Gallier calls these double cones *projective polyhedra* [5]. In the following we shall make use of his nomenclature. Note that the transition from the projective polyhedron $\tilde{\mathbf{P}} = \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\end{pmatrix}, \mathbf{0}\right) \cup \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\end{pmatrix}, \mathbf{0}\right) \subseteq \mathbb{K}^{d+1}$ to $\mathbf{P}(A, \mathbf{a}) \cup \mathbf{P}(-A, -\mathbf{a}) \subseteq \mathbb{K}^d$ geometrically corresponds to the intersection of $\tilde{\mathbf{P}}$ with the hyperplane $\left\{\binom{\mathbf{x}}{\lambda} \mid \lambda = 1\right\} \subseteq \mathbb{K}^{d+1}$.

Lemma 6 *Any linear map $\phi : \mathbb{K}^{d+1} \to \mathbb{K}^{d'+1}$ is compatible with projective equality $=_p$.*

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathrm{Proj}^d(\mathbb{K})$ with $\mathbf{u} =_p \mathbf{v}$, i.e., there exists some $\lambda \neq 0$ with $\mathbf{u} = \lambda\mathbf{v}$. Then $\phi(\mathbf{u}) = \phi(\lambda\mathbf{v}) = \lambda\phi(\mathbf{v})$, hence $\phi(\mathbf{u}) =_p \phi(\mathbf{v})$. $\square$

Definition 7 *Let $\phi : \mathbb{K}^{d+1} \to \mathbb{K}^{d+1}$ be a bijective linear map. Then ϕ is called a* projectivity*.*

Proposition 8 *The class of projective polyhedra is closed under projectivities. It is also closed under linear maps $\phi : \mathbb{K}^{d+1} \to \mathbb{K}^{d'+1}$.*

Proof. The proof is obvious since the image of a closed convex cone is a closed convex cone again. $\square$

In case of a projectivity ϕ let M be its transformation matrix. Then we can explicitly state the following formula for the image of a projective polyhedron $\tilde{\mathbf{P}} = \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\end{pmatrix}, \mathbf{0}\right) \cup \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\end{pmatrix}, \mathbf{0}\right)$:

$$M\tilde{\mathbf{P}} = \mathbf{P}\left(\begin{pmatrix}A & -\mathbf{a}\end{pmatrix}M^{-1}, \mathbf{0}\right) \cup \mathbf{P}\left(\begin{pmatrix}-A & \mathbf{a}\end{pmatrix}M^{-1}, \mathbf{0}\right).$$

Notes on Projective Polyhedra. During our research we found the following interesting properties of projective polyhedra:

(i) The class of projective polyhedra is not closed under intersections [5].

(ii) There is a close relationship between projective polyhedra and the set of feasible solutions of the linear fractional program

$$\text{maximize } \frac{\mathbf{c}^T\mathbf{x} + \alpha}{\mathbf{d}^T\mathbf{x} + \beta} \text{ subject to } A\mathbf{x} \leq \mathbf{a}.$$

Under the regularity conditions that $\{\mathbf{x} \mid A\mathbf{x} \le \mathbf{a}\}$ is bounded and not empty, Charnes and Cooper [2] show that it suffices to solve the two linear programs

$$\begin{aligned} &\text{maximize } \mathbf{c}^T\mathbf{y} + \alpha\lambda \text{ subject to} \\ &\qquad A\mathbf{y} - \mathbf{a}\lambda \le \mathbf{0}, \mathbf{d}^T\mathbf{y} + \beta\lambda = 1, \lambda \ge 0 \\ &\text{maximize } \mathbf{c}^T\mathbf{y} + \alpha\lambda \text{ subject to} \\ &\qquad -A\mathbf{y} + \mathbf{a}\lambda \le \mathbf{0}, \mathbf{d}^T\mathbf{y} + \beta\lambda = 1, \lambda \le 0. \end{aligned}$$

It is not hard to see that the intersection of the projective embedding $\mathbf{P}\left(\left(A, -\mathbf{a}\right), \mathbf{0}\right) \cup \mathbf{P}\left(\left(-A, \mathbf{a}\right), \mathbf{0}\right)$ of $\mathbf{P}\left(A, \mathbf{a}\right)$ with the hyperplane $\{\binom{\mathbf{x}}{\lambda} \mid \binom{\mathbf{d}}{\beta}^T \binom{\mathbf{x}}{\lambda} = 1\}$ corresponds to the set of feasible solutions of the linear programs above. Moreover, projective polyhedra allow us to drop the regularity conditions.

In the remainder of this section we return to general, non-polyhedral complete projective embeddings.

Proposition 9 *Let $\mathbf{Q} = \left\{\mathbf{x} \,\middle|\, \mathbf{x}^T Q\mathbf{x} \le c^2\right\}$ be a quadric. Further, let*

$$\tilde{\mathbf{Q}} = \left\{ \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} \,\middle|\, \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix}^T \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0}^T & -c^2\end{pmatrix} \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} \le 0 \right\}.$$

Then $\tilde{\mathbf{Q}}$ is the complete projective embedding of $\mathbf{Q}$.

Proof. Obviously, $\tilde{\mathbf{Q}}$ includes $\iota(\mathbf{Q})$. Furthermore, $\binom{\mathbf{x}}{\lambda}^T \left(\begin{smallmatrix} Q & \mathbf{0} \\ \mathbf{0}^T & -c^2\end{smallmatrix}\right) \binom{\mathbf{x}}{\lambda} \le 0$ implies $\mu\binom{\mathbf{x}}{\lambda}^T \left(\begin{smallmatrix} Q & \mathbf{0} \\ \mathbf{0}^T & -c^2\end{smallmatrix}\right) \mu\binom{\mathbf{x}}{\lambda} \le 0$ for all $\mu \in \mathbb{K}$. Hence, $\tilde{\mathbf{Q}}$ is projectively complete.

We show that $\tilde{\mathbf{Q}}$ is closed. Let $\left(\binom{\mathbf{x}_i}{\lambda_i}\right)_{i\in\mathbb{N}}$ be a convergent sequence with $\binom{\mathbf{x}_i}{\lambda_i} \in \tilde{\mathbf{Q}}$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \binom{\mathbf{x}_i}{\lambda_i} = \binom{\mathbf{x}}{\lambda}$. Assume, $\lambda \ne 0$. Let J be the set of all indices where $\lambda_i \ne 0$. Then $\left(\binom{\frac{1}{\lambda_i}\mathbf{x}_i}{1}\right)_{i\in J}$ is an infinite sequence which converges to $\binom{\frac{1}{\lambda}\mathbf{x}}{1}$. Furthermore, for all $i \in J$ we have $\frac{1}{\lambda_i} \in \mathbf{Q}$. Since $\mathbf{Q}$ is closed, we have $\frac{1}{\lambda}\mathbf{x} \in \mathbf{Q}$. Hence, $\binom{\frac{1}{\lambda}\mathbf{x}}{1} \in \iota(\mathbf{Q})$ and, by projective completeness, $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{Q}}$. Assume, $\lambda = 0$. Either we have an infinite subset $J \subseteq \mathbb{N}$ such that $\mathbf{x}_i^T Q\mathbf{x}_i \le 0$ for all $i \in J$. The set $\left\{\mathbf{y} \,\middle|\, \mathbf{y}^T Q\mathbf{y} \le 0\right\} \subseteq \mathbf{Q}$ is closed. Hence, $\mathbf{x}^T Q\mathbf{x} \le 0$ and $\binom{\mathbf{x}}{0}^T \left(\begin{smallmatrix} Q & \mathbf{0} \\ \mathbf{0}^T & -c^2\end{smallmatrix}\right) \binom{\mathbf{x}}{0} \le 0$. That is, $\binom{\mathbf{x}}{0} \in \tilde{\mathbf{Q}}$. Otherwise, there exists an index i_0 with $\lambda_i \ne 0$ for all $i \ge i_0$. The remaining infinite sequence $\left(\binom{\mathbf{x}_i}{\lambda_i}\right)_{i\ge i_0}$ has the same limes and without loss of generality we may assume that $i = 0$. Setting $\mathbf{y}_i = \mathbf{x}_i - \frac{\lambda_i}{\lambda_0}\mathbf{x}_0$ yields the equalities $\binom{\mathbf{y}_i}{0} = \binom{\mathbf{x}_i}{\lambda_i} - \frac{\lambda_i}{\lambda_0}\binom{\mathbf{x}_0}{\lambda_0}$ for all $i \in \mathbb{N}$. Since $\lim_{i\to\infty} \lambda_i = 0$ and $\lim_{i\to\infty} \mathbf{x}_i = \mathbf{x}$, we obtain $\lim_{i\to\infty} \frac{\lambda_i}{\lambda_0}\mathbf{x}_0 = \mathbf{0}$ and $\lim_{i\to\infty} \binom{\mathbf{y}_i}{0} = \lim_{i\to\infty} \binom{\mathbf{x}_i}{\lambda_i} - \lim_{i\to\infty} \frac{\lambda_i}{\lambda_0}\binom{\mathbf{x}_0}{\lambda_0} = \binom{\mathbf{x}}{0}$. Hence, $\binom{\mathbf{y}_i}{0}$ is a sequence with $\lim_{i\to\infty} \binom{\mathbf{y}_i}{0} = \binom{\mathbf{x}}{\lambda}$ where the last coefficient of the members is equal to zero. For this case we already have shown that $\binom{\mathbf{x}}{0} \in \tilde{\mathbf{Q}}$.

It remains to show that $\tilde{\mathbf{Q}}$ is the least closed projectively complete set which contains $\iota(\mathbf{Q})$. Assume $\tilde{\mathbf{Q}}'$ is another closed projectively complete set that contains $\iota(\mathbf{Q})$. We have to show that any $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{Q}}$ is also in $\tilde{\mathbf{Q}}'$. Therefore, let $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{Q}}$, i. e., $\mathbf{x}^T Q\mathbf{x} \le \lambda^2 c^2$. In the case $\lambda \ne 0$ we have $\frac{1}{\lambda}\mathbf{x} \in \mathbf{Q}$ and, hence $\binom{\frac{1}{\lambda}\mathbf{x}}{1} \in \iota(\mathbf{Q})$. The set $\tilde{\mathbf{Q}}'$ includes $\iota(\mathbf{Q})$ and is projectively complete. Hence, $\binom{\mathbf{x}}{\lambda} \in \tilde{\mathbf{Q}}'$. Otherwise, we have $\lambda = 0$, i. e., $\mathbf{x}^T Q\mathbf{x} \le 0$. Then for any μ we have $\mu\mathbf{x}^T Q\mu\mathbf{x} \le 0$. Let $\mu_i = i$. Since $0 \le c^2$, we have $\mu_i\mathbf{x} \in \mathbf{Q}$ for all $i \ge 1$. Further, since $\iota(\mathbf{Q}) \in \tilde{\mathbf{Q}}'$ and $\tilde{\mathbf{Q}}'$ is projectively complete, we also have $\frac{1}{\mu_i}\binom{\mu_i\mathbf{x}}{1} = \binom{\mathbf{x}}{\frac{1}{\mu_i}} \in \tilde{\mathbf{Q}}'$. We obtain $\lim_{i\to\infty} \binom{\mathbf{x}}{\frac{1}{\mu_i}} = \binom{\mathbf{x}}{0} \in \mathbb{K}^{d+1}$. Since $\tilde{\mathbf{Q}}'$ is closed, it follows $\binom{\mathbf{x}}{\lambda} = \binom{\mathbf{x}}{0} \in \tilde{\mathbf{Q}}'$. □

4 Inner Approximation of a Quadric

In the following we show how to find an inner polyhedral approximation of a quadric $\mathbf{Q} = \left\{\mathbf{x} \,\middle|\, \mathbf{x}^T Q\mathbf{x} \le c^2\right\} \subseteq \mathbb{E}^d$. Without loss of generality we may assume that Q is a symmetric matrix.[2] We will show that it suffices to provide an inscribed polyhedron of the unit sphere in any dimension to find an inner approximation of the quadric. For example, for any dimension d the polyhedron

$$\mathbf{B}(d) = \mathbf{P}\left(\left(\begin{smallmatrix}\mathrm{I}\\ -\mathrm{I}\end{smallmatrix}\right), \tfrac{1}{\sqrt{d}}\left(\begin{smallmatrix}\mathbf{1}\\ \mathbf{1}\end{smallmatrix}\right)\right) = \left\{\mathbf{x} \,\middle|\, \mathbf{x} \le \tfrac{1}{\sqrt{d}}\mathbf{1}, -\mathbf{x} \le \tfrac{1}{\sqrt{d}}\mathbf{1}\right\}$$

is an inscribed hypercube of the hyperball

$$\mathbf{S}^d = \left\{\mathbf{x} \,\middle|\, \mathbf{x}^T\mathbf{x} \le 1\right\} = \left\{\mathbf{x} \,\middle|\, x_1^2 + x_2^2 + \cdots + x_d^2 \le 1\right\}.$$

Proposition 10 (Principal Axis Transformation) *Let $\mathbf{Q} = \left\{\mathbf{x} \,\middle|\, \mathbf{x}^T Q\mathbf{x} \le c^2\right\}$ be a quadric in $\mathbb{E}^d$ where Q is a symmetric matrix. Further, let*

$$\tilde{\mathbf{Q}} = \left\{ \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} \,\middle|\, \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix}^T \tilde{Q} \begin{pmatrix}\mathbf{x}\\ \lambda\end{pmatrix} \le 0 \right\} \text{with } \tilde{Q} = \begin{pmatrix} Q & \mathbf{0} \\ \mathbf{0}^T & -c^2\end{pmatrix}$$

be the complete projective embedding of $\mathbf{Q}$ in $\mathbb{E}^{d+1}$. Then there exists an invertible matrix L and a diagonal matrix

$$E = \left(\begin{smallmatrix} \mathrm{I} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & -\mathrm{I}\end{smallmatrix}\right) \quad \textit{such that} \quad \tilde{Q} = LEL^T.$$

While the matrix L is in general not unique, the matrix E is uniquely determined by Sylvester's law of inertia. If Q is positive semidefinite, then E has exactly one negative entry.

Proof. Symmetric Gaussian elimination yields an invertible diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_{d+1}) = U\tilde{Q}U^T$ where U represents the row operations and the

[2] Otherwise, let $\hat{Q} = \frac{1}{2}(Q + Q^T)$, i. e., $\mathbf{x}^T\hat{Q}\mathbf{x} = \frac{1}{2}(\mathbf{x}^T Q\mathbf{x} + \mathbf{x}^T Q^T\mathbf{x}) = \frac{1}{2}(\mathbf{x}^T Q\mathbf{x} + (\mathbf{x}^T Q^T\mathbf{x})^T) = \frac{1}{2}(\mathbf{x}^T Q\mathbf{x} + \mathbf{x}^T Q\mathbf{x}) = \mathbf{x}^T Q\mathbf{x}$ for all $\mathbf{x}$, and use $\hat{Q}$ instead of Q.

transposed matrix U^T represents the corresponding column operations. We normalize the elements of D by multiplying with the matrix $S = \operatorname{diag}(s_1, \ldots, s_{d+1}) = S^T$, where each s_i is defined as $s_i = 1$ if $d_i = 0$, or $s_i = \frac{1}{\sqrt{|d_i|}}$ if $d_i \neq 0$. Then $SDS^T = SU\tilde{Q}U^T S^T$ is a diagonal matrix whose entries are 1, -1 or 0. Finally, we sort the entries of the diagonal matrix SDS^T in descending order, yielding a row permutation matrix P and a corresponding column permutation matrix P^T and $E = \left(\begin{smallmatrix} \mathrm{I} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} & -\mathrm{I} \end{smallmatrix}\right) = PSDS^TP^T = PSU\tilde{Q}U^TS^TP^T$. Since P, S, and U are invertible, we define $L = (PSU)^{-1}$ and obtain the factorization $\tilde{Q} = LEL^T$. □

Note that the decomposition $\tilde{Q} = LEL^T$ as given above is only possible in a projective vector space. In a non-projective setting we would obtain a richer variety of resulting forms.

Let $\mathbf{Q}$, $\tilde{\mathbf{Q}}$, Q, $\tilde{Q}$, and L, E be given as in Prop. 10. We define $\binom{\mathbf{y}}{\mu} = L^T\binom{\mathbf{x}}{\lambda}$ and obtain the identity $\mathbf{x}^TQ\mathbf{x} - \lambda^2c^2 = \binom{\mathbf{x}}{\lambda}^T\tilde{Q}\binom{\mathbf{x}}{\lambda} = \binom{\mathbf{x}}{\lambda}^T LEL^T\binom{\mathbf{x}}{\lambda} = \binom{\mathbf{y}}{\mu}^T E\binom{\mathbf{y}}{\mu}$. Hence, the base transformation matrix L transforms the projective quadric $\tilde{\mathbf{Q}}$ into a projective hyperboloid of the form $\tilde{\mathbf{H}} = \left\{\binom{\mathbf{y}}{\mu} \,\middle|\, \binom{\mathbf{y}}{\mu}^T E\binom{\mathbf{y}}{\mu}\right\}$.

We explain how to find an inner approximation of the projective hyperboloid $\tilde{\mathbf{H}}$. Application of the inverse base transformation finally yields (a union of two) inscribed polyhedra of $\mathbf{Q}$. Let k the number of 1s, l the numbers of 0s, and m the number of -1s in E, with $k + l + m = d + 1$. According to Sylvester's law of inertia, k, l, and m are uniquely determined by Q.

We characterize the solutions $\binom{\mathbf{y}}{\mu}$ of $\binom{\mathbf{y}}{\mu}^T E\binom{\mathbf{y}}{\mu} \leq 0$, or equivalently:

$$y_1^2 + \cdots + y_k^2 - y_{k+l+1}^2 - \cdots - y_d^2 - \mu^2 \leq 0. \qquad (2)$$

We have the following mutual exclusive cases:

(i) If $k = 0$, then $\tilde{\mathbf{H}} = \mathbb{E}^{d+1}$, i. e., $\tilde{\mathbf{H}}$ is the complete projective embedding of the entire vector space $\mathbb{E}^d$.

(ii) If $k = d+1$, then $\tilde{\mathbf{H}} = \{\mathbf{0}\}$, i. e., $\tilde{\mathbf{H}}$ is the complete projective embedding of the empty set.

(iii) If $0 < k < d+1$ and $m = 0$, then $\tilde{\mathbf{H}} = \{0\}^k \times \mathbb{E}^{d-k} \times \mathbb{E}$, i. e., $\tilde{\mathbf{H}}$ is the complete projective embedding of the subspace $\{0\}^k \times \mathbb{E}^{d-k}$.

(iv) If $0 < k < d + 1$ and $m = 1$, then $\tilde{\mathbf{H}} = \{\binom{\mathbf{y}}{\mu} \mid \mathbf{y}^T\left(\begin{smallmatrix} \mathrm{I} & \mathrm{O} \\ \mathrm{O} & \mathrm{O} \end{smallmatrix}\right)\mathbf{y} \leq \mu^2\}$ is the complete projective embedding of the cylinder $\mathbf{S}^k \times \mathbb{E}^{d-k}$. Let $\mathbf{T}(k) = \mathbf{P}(A, \mathbf{a})$ be an inscribed polyhedron of $\mathbf{S}^k$. The cylindrification $\mathbf{T}(k) \times \mathbb{E}^{d-k}$ of $\mathbf{T}(k)$ is given by $\mathbf{P} = \mathbf{P}\left(\begin{pmatrix} A & \mathrm{O} \end{pmatrix}, \mathbf{a}\right)$. We have $\mathbf{P} \subseteq \mathbf{S}^k \times \mathbb{E}^{d-k}$. According to Prop. 3 the subset relation carries over to the complete projective embeddings $\tilde{\mathbf{P}} \subseteq \tilde{\mathbf{H}}$, where $\tilde{\mathbf{P}} = \mathbf{P}\left(\left(\begin{smallmatrix} A & \mathrm{O} & -\mathbf{a} \\ \mathbf{0}^T & \mathbf{0}^T & -1 \end{smallmatrix}\right), \binom{\mathbf{0}}{0}\right) \cup \mathbf{P}\left(\left(\begin{smallmatrix} -A & \mathrm{O} & \mathbf{a} \\ \mathbf{0}^T & \mathbf{0}^T & 1 \end{smallmatrix}\right), \binom{\mathbf{0}}{0}\right)$. Hence, after application of the base transformation L^{-T} on $\tilde{\mathbf{P}}$ we obtain a union of polyhedra $L^T(\tilde{\mathbf{P}}) = \mathbf{P}\left(\begin{pmatrix} A_1 & -\mathbf{a}_1 \end{pmatrix}, \mathbf{0}\right) \cup \mathbf{P}\left(\begin{pmatrix} -A_1 & \mathbf{a}_1 \end{pmatrix}, \mathbf{0}\right)$ with $L^T(\tilde{\mathbf{P}}) \subseteq \tilde{\mathbf{Q}}$ and the matrices are given by $\begin{pmatrix} A_1 & -\mathbf{a}_1 \end{pmatrix} = \left(\begin{smallmatrix} A & \mathrm{O} & -\mathbf{a} \\ \mathbf{0}^T & \mathbf{0}^T & -1 \end{smallmatrix}\right)L^T$ and $\begin{pmatrix} -A_1 & \mathbf{a}_1 \end{pmatrix} = \left(\begin{smallmatrix} -A & \mathrm{O} & \mathbf{a} \\ \mathbf{0}^T & \mathbf{0}^T & 1 \end{smallmatrix}\right)L^T$. According to Prop. 5, $L^T(\tilde{\mathbf{P}})$ is the complete projective embedding of $\mathbf{R} = \mathbf{P}(A_1, \mathbf{a}_1) \cup \mathbf{P}(-A_1, -\mathbf{a}_1)$. Furthermore, by Prop. 3 we have $\mathbf{R} \subseteq \mathbf{Q}$.

(v) Otherwise, we have $0 < k < d+1$ and $m > 1$. Now, let E' be the matrix which is obtained by replacing all but the last occurrence of -1 in E by 0, i. e., $E' = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0, -1)$. Any solution of $\binom{\mathbf{y}}{\mu}^T E'\binom{\mathbf{y}}{\mu} \leq 0$ is also a solution of (2), since for all $y_1, \ldots, y_d$ it holds $y_1^2 + \cdots + y_k^2 \geq y_1^2 + \cdots + y_k^2 - y_{k+l}^2 - \cdots - y_d^2$. Hence, $\tilde{\mathbf{H}}' = \left\{\binom{\mathbf{y}}{\mu} \,\middle|\, \binom{\mathbf{y}}{\mu}^T E'\binom{\mathbf{y}}{\mu} \leq 0\right\}$ is the complete projective embedding of the cylinder $\mathbf{S}^k \times \mathbb{E}^{d-k}$ and $\tilde{\mathbf{H}}' \subseteq \tilde{\mathbf{H}}$. Now, we proceed like in the previous item, where we use E' and $\tilde{\mathbf{H}}'$ instead of E and $\tilde{\mathbf{H}}$.

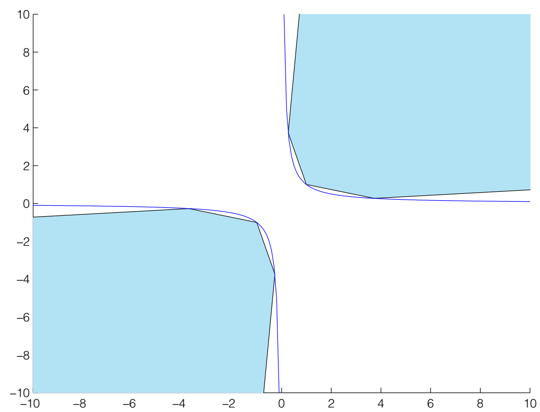

Figure 1: Hexagon Inscribed in Hyperbola $xy \geq 1$

5 Generating Polytopes With Circumsphere

In this section we discuss the problem of generating polytopes with circumsphere of arbitrary dimensions.

Obviously, the convex hull of several sampling points $\mathbf{v}_1, \ldots, \mathbf{v}_n$ on the boundary of the d-dimensional hyperball $\mathbf{S}$ results in an inscribed $\mathcal{V}$-polytope $\mathbf{P}$. Hence, to obtain the corresponding $\mathcal{H}$-representation, we had to perform a costly facet-enumeration and lose control on the number of facets of the resulting $\mathcal{H}$-representation. Instead, we are interested in methods which allow us to generate inscribed $\mathcal{H}$-polytopes directly. We abandon the idea of taking samples and provide a more regular way to generate inscribed $\mathcal{H}$-polyhedra. An interesting class of polytopes with circumsphere is the class of uniform polytopes [4, 3] which includes the class of regular polytopes. A complete enumeration of uniform polyhedra is only known in low dimensions. In two dimensions the uniform polytopes are the infinitely many regular polygons. In three dimensions the uniform polytopes cover the five Platonic solids, the 13 Archimedean solids, and the infinite set of prisms and antiprisms. In all dimensions the class of uniform polytopes contains the regular simplex, the hypercube, and the cross polytope. A d-dimensional simplex has only $d+1$ facets and vertices, which is certainly insufficient to provide a good inner approximation of a sphere. The d-dimensional cross polytope has 2^d facets and $2d$ vertices. Hence, the resulting $\mathcal{H}$-polytopes are only feasible in lower dimensions. The dual of a d-dimensional cross polytope is the d-dimensional hypercube having $2d$ facets and 2^d vertices. Apparently, the hypercube is well suited for computational purposes, but still may have too less facets to provide a good inner approximation of the sphere.

Hence, it is desirable to have a regular method which allows us to generate a richer variety of uniform polytopes. The next proposition provides such a method.

Proposition 11 *Given a polytope $\mathbf{P}_1 = \mathbf{P}(A_1, \mathbf{a}_1) \in \mathbb{K}^{d_1}$ with unit circumsphere $\mathbf{S}^{d_1}$ and a polytope $\mathbf{P}_2 = \mathbf{P}(A_2, \mathbf{a}_2) \in \mathbb{K}^{d_2}$ with unit circumsphere $\mathbf{S}^{d_2}$. Then for any $\alpha > 0$, $\beta > 0$ with $\alpha^2 + \beta^2 = 1$ the weighted cross product*

$$\begin{aligned}\mathbf{P} = \alpha\mathbf{P}_1 \times \beta\mathbf{P}_2 &= \{ \left(\begin{smallmatrix}\mathbf{x}\\ \mathbf{y}\end{smallmatrix}\right) \,|\, \mathbf{x} \in \alpha\mathbf{P}_1,\, \mathbf{y} \in \beta\mathbf{P}_2 \} \\ &= \mathbf{P}\left(\left(\begin{smallmatrix} A_1 & \mathrm{O} \\ \mathrm{O} & A_2 \end{smallmatrix}\right), \left(\begin{smallmatrix}\alpha\mathbf{a}_1 \\ \beta\mathbf{a}_2\end{smallmatrix}\right) \right)\end{aligned}$$

is a polytope in $\mathbb{K}^{d_1+d_2}$ with unit circumsphere $\mathbf{S}^{d_1+d_2}$. Furthermore, let f_1, f_2 be the number of facets of $\mathbf{P}_1$ and $\mathbf{P}_2$, and let v_1, v_2 be the number of vertices of $\mathbf{P}_1$ and $\mathbf{P}_2$. Then $\mathbf{P}$ has $f_1 + f_2$ facets and $v_1 v_2$ vertices.

Proof. The statement on the number of vertices and facets is rather obvious and belongs to mathematical folklore. We show that $\mathbf{P}$ has the circumsphere $\mathbf{S}^{d_1+d_2}$. That is, for any vertex $\left(\begin{smallmatrix}\alpha\mathbf{x}_i\\ \beta\mathbf{y}_j\end{smallmatrix}\right)$, $i = 1, \dots, v_1$, $j = 1, \dots, v_2$ of $\mathbf{P}$ we have $\left|\left(\begin{smallmatrix}\alpha\mathbf{x}_i\\ \beta\mathbf{y}_j\end{smallmatrix}\right)\right| = \left(\begin{smallmatrix}\alpha\mathbf{x}_i\\ \beta\mathbf{y}_j\end{smallmatrix}\right)^T \left(\begin{smallmatrix}\alpha\mathbf{x}_i\\ \beta\mathbf{y}_j\end{smallmatrix}\right) = \alpha^2 \mathbf{x}_i^T \mathbf{x}_i + \beta^2 \mathbf{y}_j^T \mathbf{y}_j = \alpha^2 + \beta^2 = 1$. $\square$

Clearly, the previous proposition can be generalized to the weighted cross product of finitely many circumscribed polytopes. For example, the d-dimensional hypercube is the d-fold weighted cross product of the line segments $[-1, 1]$ with weight $\alpha = \frac{1}{\sqrt{d}}$.

6 Conclusion

We discussed a projective generalization of closed sets which coincides in the case of polyhedral closed set with Gallier's projective polyhedra and the set of feasible solutions of a linear fractional program. Within this framework we used projective base transformation to generate inscribed $\mathcal{H}$-polyhedra of arbitrary quadrics provided we have given inscribed $\mathcal{H}$-polyhedra of the unit sphere. Finally, we discussed an easy method to generate inscribed polyhedra of a higher dimensional sphere out of inscribed polyhedra of spheres of lower dimension.

We have not discussed any quantifiable predication on the quality of the approximation, like ratio of the volumes or Hausdorff distance of both sets. Although it would be interesting to have some measure for the quality of the approximation, both notions are not well-defined for unbounded sets which may occur in our setting. However, for our purpose – finding inner approximation of a level set – the presented approach turned out to work well and there was no need to extend our very limited selection of template polyhedra with circumsphere.

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