1-String B_1-VPG Representations of Planar Partial 3-Trees and Some Subclasses

Therese Biedl* Martin Derka†

Abstract

Planar partial 3-trees are subgraphs of those planar graphs obtained by repeatedly inserting a vertex of degree 3 into a face. In this paper, we show that planar partial 3-trees have 1-string B_1-VPG representations, i.e., representations where every vertex is represented by an orthogonal curve with at most one bend, every two curves intersect at most once, and intersections of curves correspond to edges in the graph. We also show that some subclasses of planar partial 3-trees have {`L`}-representations, i.e., a B_1-VPG representation where every curve has the shape of an `L`.

1 Introduction

A *string representation* is a representation of a graph where every vertex v is assigned a curve $\mathbf{v}$. Vertices u, v are connected by an edge if and only if curves $\mathbf{u}, \mathbf{v}$ intersect. A 1-*string representation* is a string representation where every two curves intersect at most once.

String representations of planar graphs were first investigated by Ehrlich, Even and Tarjan in 1976 [12]. They showed that every planar graph has a 1-string representation using "general" curves. In 1984, Scheinerman conjectured [18] that every planar graph has a 1-string representation, and furthermore curves are line segments (not necessarily axis-parallel). Chalopin, Gonçalves and Ochem [7, 8] proved that every planar graph has a 1-string representation in 2007. Scheinerman's conjecture itself remained open until 2009 when it was proved true by Chalopin and Gonçalves [6].

Our paper investigates string representations that use *orthogonal curves*, i.e., curves consisting of vertical and horizontal segments. If every curve has at most k bends, these are called B_k-*VPG representations*. The hierarchy of B_k-VPG representations was introduced by Asinowski et al. [1, 2]. VPG is an acronym for Vertex-Path-Grid since vertices are represented by paths in a rectangular grid.

It is easy to see that all planar graphs are VPG-graphs (e.g. by generalizing the construction of Ehrlich, Even and Tarjan). For bipartite planar graphs, curves can even be required to have no bends [17, 11]. For arbitrary planar graphs, bends in orthogonal curves are required. Chaplick and Ueckerdt showed that 2 bends per curve always suffice [10]. In a recent paper we strengthened this to give a B_2-VPG representation that is also a 1-string representation [3].

B_k-VPG representations were further studied by Chaplick, Jelínek, Kratochvíl and Vyskočil [9] who showed that recognizing B_k-VPG graphs is NP-complete even when the input graph is given by a B_{k+1}-VPG representation, and that for every k, the class of B_{k+1}-VPG graphs is strictly larger than B_k-VPG.

Our Contribution Felsner et al. [14] showed that every planar 3-tree has a B_1-VPG representation. Moreover, every vertex-curve has the shape of an `L` (we call this an {`L`}*-representation*). This implies that any two vertex-curves intersect at most once, so this is a 1-string B_1-VPG representation. In this paper, we extend the result to more graphs, and in particular, show:

Theorem 1 *Every planar partial 3-tree G has a 1-string B_1-VPG representation.*

There are 4 possible shapes of orthogonal curves with one bend. Depending on where the bend is situated, we call them `L`, `Γ`, `ℸ` and `⅃` respectively. Note that a horizontal or vertical curve without bends can be turned into any of the shapes by adding one bend.

The construction of our proof of Theorem 1 uses all 4 possible shapes `L`, `Γ`, `ℸ`, `⅃`. However, for some subclasses of planar partial 3-trees, we can show that fewer shapes suffice. We use the notation {`L`, `ℸ`}-representation for a B_1-VPG representation where all curves are either `L` or `ℸ`, and similarly for other subsets of shapes. We can show the following:

Theorem 2 *Any IO-graph has an* {`L`}*-representation.*

Theorem 3 *Any Halin-graph has an* {`L`, `ℸ`}*-representation, and only one vertex uses a* `ℸ`*-shape.*

We give the definitions of these graph classes and the proof of these theorems in the next three sections, and end with open problems in Section 5.

*David R. Cheriton School of Computer Science, University of Waterloo, biedl@uwaterloo.ca. Research supported by NSERC.

†David R. Cheriton School of Computer Science, University of Waterloo, mderka@uwaterloo.ca. Research supported by an NSERC Vanier CGS.

2 Planar partial 3-trees

A *planar graph* is a graph that can be drawn without edge crossings. If one such drawing Γ is fixed, then a *face* is a maximal connected region of $\mathbb{R}^2 - \Gamma$. The *outer face* corresponds to the unbounded region; the *interior faces* are all other faces. A vertex is called *exterior* if it is on the outer face and *interior* otherwise.

A 3-*tree* is a graph that is either a triangle or has a vertex order $v_1, \ldots, v_n$ such that for $i \geq 4$, vertex v_i is adjacent to exactly three predecessors and they form a triangle. A *partial* 3-*tree* is a subgraph of a 3-tree.

Our proof of Theorem 1 employs the method of "private regions" used previously for various string representation constructions [3, 8, 14]. We define the following:

Definition 1 (F-shape and rectangular shape) *An* F-shaped area *is a region bounded by a 10-sided polygon with CW or CCW sequence of interior angles* $90°$, $270°$, $90°$, $90°$, $270°$, $270°$, $90°$, $90°$, $90°$ *and* $90°$. *A* rectangle-shaped area *is a region bounded by an axis-aligned rectangle.*

Definition 2 (Private region) *Given a* 1*-string representation, a* private region *of vertices* $\{a, b, c\}$ *is an F-shaped or rectangle-shaped area that intersects (up to permutation of names) curves* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *in the way depicted in Figure 1(a), and that intersects no other curves and private regions.*

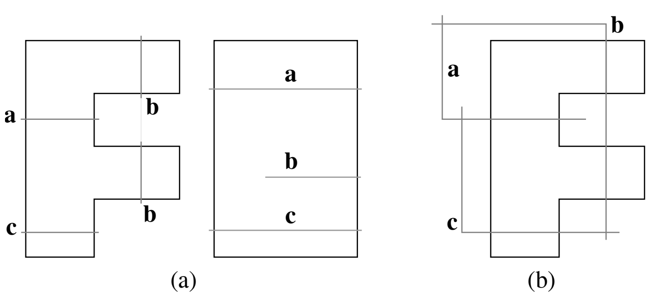

Figure 1: (a) An F-shaped (left) and rectangle-shaped (right) private region of $\{a, b, c\}$. (b) The base case. Intersections among $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ can be omitted as needed.

Now we are ready to prove Theorem 1. Let G be a planar partial 3-tree. By definition, there exists a 3-tree H for which G is a subgraph. One can show [4] that we may assume H to be planar. Let $v_1, \ldots, v_n$ be a vertex order of H such that for $i \geq 4$ vertex v_i is adjacent to 3 predecessors that form a triangle. In particular, v_4 is incident to a triangle formed by $\{v_1, v_2, v_3\}$. One can show (see e.g. [4]) that the vertex order can be chosen in such a way that $\{v_1, v_2, v_3\}$ is the outer face of H in some planar drawing.

For $i \geq 3$, let G_i and H_i be the subgraphs of G (respectively H) induced by vertices $v_1, \ldots, v_i$. We prove Theorem 1 by showing the following by induction on i:

> G_i has a 1-string B_1-VPG representation with a private region for every interior face of H_i.

In the base case, $i = 3$ and $G \subseteq K_3 \simeq H$. Construct a representation R and find a private region for the unique interior face of H as depicted in Figure 1(b).

Now consider $i \geq 4$. By induction, construct a representation R_0 of G_{i-1} that contains a private region for every interior face of H_{i-1}.

Let $\{a, b, c\}$ be the predecessors of v_i in H. Recall that they form a triangle. Since H is planar, this triangle must form a face in H_{i-1}. Since $\{v_1, v_2, v_3\}$ is the outer face of H (and hence also of H_{i-1}), the face into which v_i is added must be an interior face, so there exists an interior face $\{a, b, c\}$ in H_{i-1}. Let P_0 be the private region that exists for $\{a, b, c\}$ in R_0; it can have the shape of an F or a rectangle.

Observe that in G, vertex v_i may be adjacent to any possible subset of $\{a, b, c\}$. This gives 16 cases (two possible shapes, up to rotation and reflection, and 8 possible adjacencies).

In each case, the goal is to place a curve $\mathbf{v_i}$ inside P_0 such that it intersects exactly the curves of the neighbours of v_i in $\{a, b, c\}$ and no other curve. Furthermore, having placed $\mathbf{v_i}$ into P_0, we need to find a private region for the three new interior faces in H_i, that is, the three faces formed by v_i and two of $\{a, b, c\}$.

Case 1: P_0 has the shape of an F. After possible rotation / reflection of R_0 and renaming of $\{a, b, c\}$ we may assume that P_0 appears as in Figure 1(a). If (v_i, a) is an edge, then place a bend for curve $\mathbf{v_i}$ in the region above $\mathbf{a}$. Let the vertical segment of $\mathbf{v_i}$ intersect $\mathbf{a}$ and (optionally) $\mathbf{c}$. Let the horizontal segment of $\mathbf{v_i}$ intersect (optionally) the top occurrence of $\mathbf{b}$. If (v_i, a) is not an edge but (v_i, c) is an edge, then place a bend for $\mathbf{v_i}$ in the region below $\mathbf{a}$, let the vertical segment of $\mathbf{v_i}$ intersect $\mathbf{c}$ and the horizontal segment of $\mathbf{v_i}$ intersect (optionally) $\mathbf{b}$. Finally, if neither (v_i, a) nor (v_i, c) is an edge, then $\mathbf{v_i}$ is a horizontal segment in the region below $\mathbf{a}$ and above $\mathbf{c}$ that (optionally) intersects $\mathbf{b}$.

In all sub-cases, $\mathbf{v_i}$ remains inside P_0, so it cannot intersect any other curve of R_0. Private regions for the newly created faces can be found as shown in Figure 2.

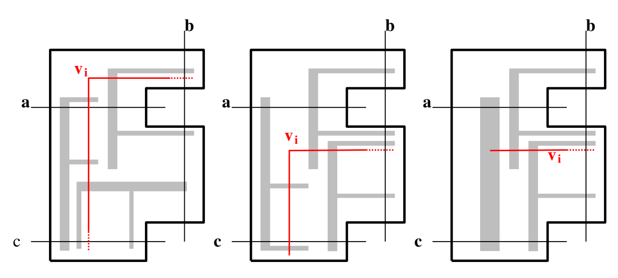

Figure 2: Inserting curve $\mathbf{v_i}$ into an F-shaped private region. (Left) (v_i, a) is an edge. (Middle) $(v_i, a) \notin E$, but $(v_i, c) \in E$. (Right) $(v_i, a), (v_i, c) \notin E$.

Case 2: P_0 has the shape of a rectangle. After possible rotation / reflection of R_0 and renaming of $\{a, b, c\}$ we may assume that P_0 appears as in Figure 1(a). If (v_i, a) is an edge, then $\mathbf{v}$ is a vertical segment that intersects $\mathbf{a}$ and (optionally) $\mathbf{b}$ and (optionally) $\mathbf{c}$. If (v_i, c) is an edge, then symmetrically $\mathbf{v_i}$ is a vertical segment that intersects $\mathbf{c}$ and (optionally) $\mathbf{b}$ and $\mathbf{a}$. Finally if neither (v_i, a) nor (v_i, c) is an edge, then let $\mathbf{v}$ be a horizontal segment between $\mathbf{a}$ and $\mathbf{c}$ with (optionally) a vertical segment attached to create an intersection with $\mathbf{b}$.

In all cases, $\mathbf{v_i}$ remains inside P_0, so it cannot intersect any other curve of R_0. Private regions for the newly created faces can be found as shown in Figure 3.

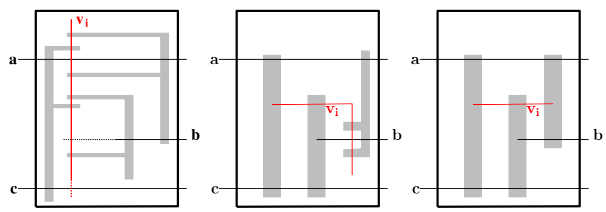

Figure 3: Inserting curve $\mathbf{v_i}$ into a rectangle-shaped private region. (Left) (v_i, a) is an edge. (Middle) $(v_i, a), (v_i, c) \notin E$, but $(v_i, b) \in E$. (Right) $(v_i, a), (v_i, b), (v_i, c) \notin E$.

Theorem 1 now holds by induction. □

We note here that in our proof-approach, both types of private regions and all four shapes with one bend are required in some cases.

3 IO-Graphs

An *IO-graph* [13] is a 2-connected planar graph with a planar embedding such that the interior vertices form a (possibly empty) independent set. One can easily show [13] that every IO-graph is a planar partial 3-tree.

We now prove Theorem 2 by constructing an $\{\mathtt{L}\}$-representation of an IO-graph G. Let O be the set of exterior vertices; by definition these induce an *outerplanar graph*, i.e., a graph that can be embedded so that all vertices are on the outer face. Moreover, since G is 2-connected, the outer face is a simple cycle, and hence the outerplanar graph $G[O]$ is also 2-connected. We first construct an $\{\mathtt{L}\}$-representation of $G[O]$, and then insert the interior vertices. To do so, we again use private regions, but we modify their definition slightly in three ways: (1) Interior vertices may have arbitrarily high degree, and so the private regions must be allowed to cross arbitrarily many curves. (2) Interior vertices may only be adjacent to exterior vertices. It therefore suffices for the private region of a face f to intersect only those curves that belong to exterior vertices on f.

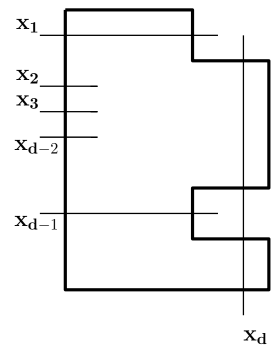

Figure 4: An IO-private region. We require that the supporting line of $\mathbf{x_i}$ (for $i = 2, \ldots, k-2$) intersects the upper segment of $\mathbf{x_d}$.

It is exactly this latter observation that allows us to find private regions more easily, therefore use fewer shapes for them, and therefore use fewer shapes for the curves. We can therefore also add: (3) The private region must be an F-shape, and it must be in the rotation Ł. The formal definition is given below:

Definition 3 (IO-private region) *Given a 1-string representation of an IO-graph, an* IO-private region *of a face f is an F-shaped area P, in the rotation Ł, which intersects curves $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_d}$ as shown in Figure 4. Here, $\{x_1, \ldots, x_d\}$ is a subset of the vertices of f enumerated in CCW order, and includes all exterior vertices that belong to f (it may or may not include other vertices). Lastly, P intersects no other curves and no other private regions.*

Lemma 4 *Any outer planar graph has an $\{\mathtt{L}\}$-representation with an IO-private region for every interior face.*

Proof. We may assume that the outerplanar graph is 2-connected, otherwise we can add vertices to make it so and delete their curves later. Enumerate the vertices on the outer face as $v_1, \ldots, v_k$ in CCW order. For every vertex v_i on the outer-face, let $\mathbf{v_i}$ be an $\mathtt{L}$ with bend at $(i, -i)$. The vertical segment of $\mathbf{v_i}$ reaches until $(i, -r_i + \varepsilon)$, where $r_i = \min\{j : (v_j, v_i) \in E\}$. (Use $r_0 = 0$.) The horizontal segment of $\mathbf{v_i}$ reaches until $(s_i + \varepsilon, i)$, where $s_i = \max\{j : (v_j, v_i) \in E\}$. (Use $s_k = k$.) See also Figure 5.

It is quite easy to see that this is a 1-string representation. For every edge (v_i, v_k) with $i < k$ we have created an intersection at $(k, -i)$. Assume for contradiction that $\mathbf{v_i}$ and $\mathbf{v_k}$ intersect for some $(v_i, v_k) \notin E$ with $i < k$. Then we must have $s = \max\{j : (v_i, v_j) \in E\} > k$, else there is no intersection. Also $r = \min\{j : (v_j, v_k) \in E\} < i$, else there is no intersection. But then $\{v_i, v_k, v_s, v_r\}$, together with the outer face, form a K_4-minor; this is impossible in an outer planar graph.

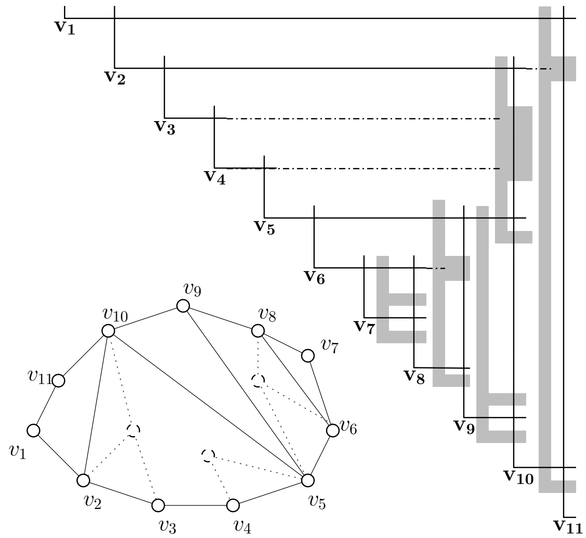

Figure 5: Example of an IO-graph and the {L}-representation of $G[O]$. The IO-private regions are shaded in grey.

Thus we found the {L}-representation. To find IO-private regions, we stretch horizontal segments of curves further as follows. For vertex v_i, set $t_i = \max\{j : v_i$ and v_j are on a common interior face$\}$. If $t_i > s_i$, then expand $\mathbf{v_i}$ horizontally until $t_i - \varepsilon$. To see that this does not introduce new crossings, observe that adding (v_i, v_{t_i}) to the graph would not destroy outerplanarity, since the edge could be routed inside the common face. The {L}-representation of such an expanded graph would contain the constructed one and also contain the added segment. Therefore the added segment cannot intersect any other curves.

After stretching all curves horizontally in this way, an IO-private region for each interior face f can then be inserted to the left of the vertical segment of $\mathbf{v_j}$, where v_j is the vertex on f with maximal index; see also Figure 5. □

Now we can prove Theorem 2, i.e., we can show that every IO-graph G has an {L}-representation. Start with the {L}-representation of $G[O]$ of Lemma 4. We add the interior vertices $v_1, \ldots, v_{n-k}$ to this in arbitrary order, maintaining the following invariant:

> For every interior face of the current graph there exists an IO-private region.

Clearly this invariant holds for the representation of $G[O]$. Let v be the next interior vertex to be added, and let f be the face where it should be inserted. By induction there exists a IO-private region P_0 for face f such that the curves $\mathbf{x_1}, \ldots, \mathbf{x_d}$ that intersect P_0 include the curves of all exterior vertices that are on f, in CCW order. We need to place an L-curve $\mathbf{v}$ into P_0, intersecting curves of neighbours of v and nothing else, and then find IO-private regions for every newly created face.

Since the interior vertices form an independent set, all neighbours of v are on the outer face, and hence belong to $\{x_1, \ldots, x_d\}$. Since G is 2-connected, v has at least two such neighbours. We have two cases.

Case 1. If (v, x_d) is not an edge, then $\mathbf{v}$ is a vertical segment that extends from the topmost to the bottommost of the curves of its neighbours, and intersects these curves after expanding them rightwards.

Since the order of $\mathbf{x_1}, \ldots, \mathbf{x_d}$ is CCW around the outer face, for every newly created face f' incident to v we have a region inside P_0 in which the curves of outer face vertices on f' appear in CCW order. IO-private regions for these faces can be found as shown in Figure 6(top). Note that some of these private regions intersect v while others do not; both are acceptable since v is on those faces, but not an exterior vertex.

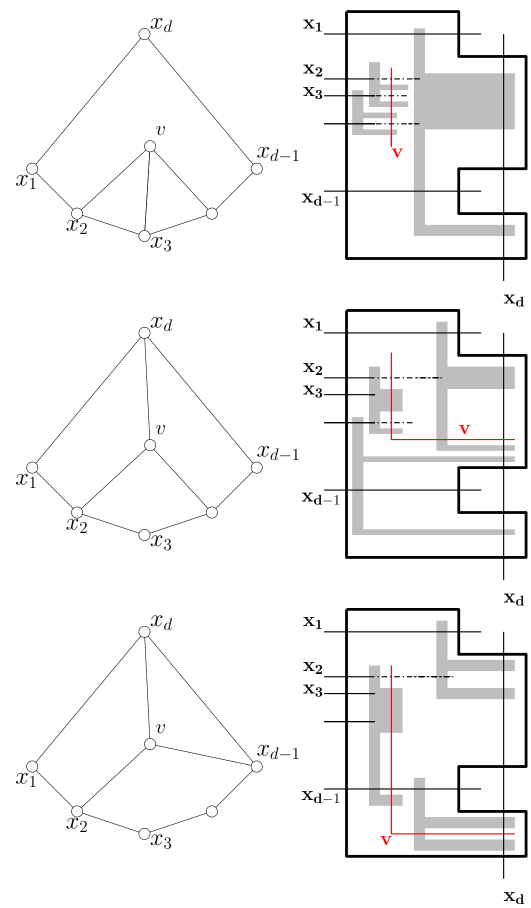

Figure 6: Inserting a vertex into a face of an IO-graph. (Top) v is not adjacent to x_d. (Middle) v is adjacent to x_d, but not x_{d-1}. (Bottom) v is adjacent to both x_d and x_{d-1}.

Case 2. If (v, x_d) is an edge, then $\mathbf{v}$ is an L, with the bend below $\mathbf{x_{d-1}}$ if (v, x_{d-1}) is an edge and above

$\mathbf{x_{d-1}}$ otherwise. The vertical segment of $\mathbf{v}$ extends from this bend to the topmost of v's neighbours in $\{\mathbf{x_1}, \dots, \mathbf{x_{d-1}}\}$, and intersects the curves of these neighbours after expanding them rightwards. The horizontal segment extends as to intersect $\mathbf{x_d}$.

IO-private regions can again be found easily, see Figure 6(middle and bottom).

Repeating this insertion operation for all interior vertices hence gives the desired representation of G. □

4 Halin graphs

A *Halin-graph* [16] is a graph obtained by taking a tree T with $n \geq 3$ vertices that has no vertex of degree 2 and connecting the leaves in a cycle. Such graphs were originally of interest since they are minimally 3-connected, but it was later shown that they are also planar partial 3-trees [5].

We now prove Theorem 3 and show that any Halin-graph G has a {⅂,L}-representation. We note here that our construction works even if T has some vertices of degree 2. Fix an embedding of G such that the outer face is the cycle C connecting the leaves of tree T. Enumerate the outer face as $v_1, \dots, v_k$ in CCW order. Since every exterior vertex was a leaf of T, vertex v_k has degree 3; let r be the interior vertex that is a neighbour of v_k. Root T at r and enumerate the vertices of T in post-order as $w_1, \dots, w_n$, starting with the leaves (which are $v_1, \dots, v_k$) and ending with r.

Let G_i be the graph induced by $w_1, \dots, w_i$. Call vertex v_j *unfinished* in G_i if it has a neighbour in $G - G_i$. For $i = k, \dots, n$, we create an {L}-representation of $G_i - (v_1, v_k)$ that satisfies the following:

> For any unfinished vertex v, curve $\mathbf{v}$ ends in a horizontal ray, and the top-to-bottom order of these rays corresponds to the CW order of the unfinished vertices on the outer face while walking from v_1 to v_k.

The {L}-representation of $G_k - (v_1, v_k)$ (i.e., the path $v_1, \dots, v_k$) is obtained easily by placing the bend for $\mathbf{v_i}$ at $(i, -i)$, giving the vertical segment length $1 + \varepsilon$ and leaving the horizontal segment as a ray as desired. To add vertex w_i for $i > k$, let $x_1, \dots, x_d$ be its children in T; their curves have been placed already. Insert a vertical segment for $\mathbf{w_i}$ with x-coordinate i, and extending from just below the lowest curve of $\mathbf{x_1}, \dots, \mathbf{x_d}$ to just above the highest. The rays of $\mathbf{x_1}, \dots, \mathbf{x_d}$ end at x-coordinate $i + \varepsilon$, while $\mathbf{w_i}$ appends a horizontal ray at its lower endpoint.

Since adding w_i means that $x_1, \dots, x_d$ are now finished (no vertex has two parents), the invariant holds. Continuing until $i = n$ yields an {L}-representation of $G - (v_1, v_k)$. It remains to add an intersection for edge (v_1, v_k). To do so, we change the shape of $\mathbf{v_1}$. Observe that its vertical segment was not used for any intersection, and that its horizontal segment can be expanded until $(n + 1, -1)$ without intersecting anything except its neighbours. After this expansion, we add a vertical segment going downward at its right end. Since v_k is a neighbour of r, curve $\mathbf{v_k}$ ended when $\mathbf{r}$ was added, i.e., at x-coordinate $n + \varepsilon$, and we can extend it until x-coordinate $n + 1 + \varepsilon$. Hence $\mathbf{v_1}$ and $\mathbf{v_k}$ can meet at $(n + 1, -k)$ if we change the shape of $\mathbf{v_1}$ to ⅂. We have hence proved Theorem 3. □

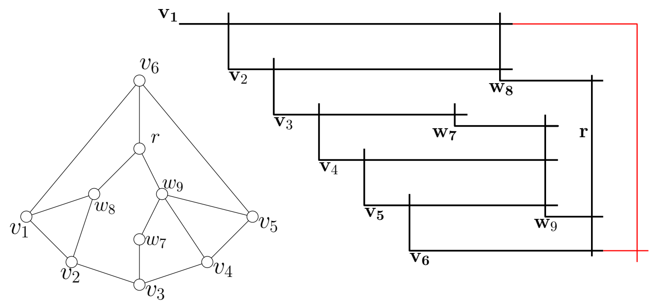

Figure 7: Example of an extended Halin-graph and its {L, ⅂}-representation, obtained by changing the curve of $\mathbf{v_1}$ so that it intersects $\mathbf{v_k}$.

Notice that in the construction for Halin-graphs, any intersection of curves occurs near the end one of the two curves. Our result therefore holds not only for Halin graphs, but also for any subgraph of a Halin graph.

The natural question to ask is whether any Halin graph has an {L}-representation, i.e., whether it is possible to avoid the single ⅂-shape that we used for $\mathbf{v_1}$. In very recent work [15] done independently from ours, Francis and Lahiri answered this question affirmatively and proved that every Halin graph has an {L}-representation.

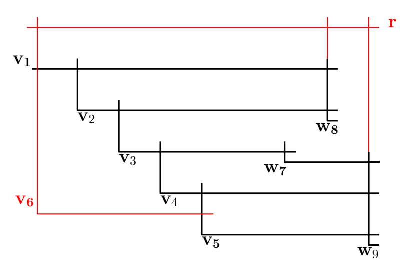

Figure 8: {L}-representation, obtained by changing the curve of $\mathbf{r}$ and $\mathbf{v_k}$, if r has no other neighbours on the outer face.

5 Conclusion

In this paper, we studied 1-string VPG-representations of planar graphs such that curves have at most one bend. It is not known whether all planar graphs have such a representation, but curiously, also no planar graph is known that does not have an {L}-representation. Felsner et al. [14] asked whether every planar graph has a {Γ, L}-representation since (as they point out) a positive answer would provide a different proof of Scheinerman's conjecture. They proved this for planar 3-trees.

In this paper, we made another step towards their question and showed that every planar partial 3-tree has a 1-string B_1-VPG representation. We also showed that IO-graphs and Halin-graphs have {L}-representations, except that for Halin-graphs one vertex curve might be a ⅂.

The obvious direction for future work is to show that all planar partial 3-trees have {L}-representations, or at least {L, Γ}-representations. As a first step, an interesting subclass would be those 2-connected planar graphs G where deleting the vertices on the outer face leaves a forest; these encompass both IO-graphs and Halin graphs.

Note that all representations constructed in this paper are *ordered*, in the sense that the order of intersections along the curves of vertices corresponds to the order of edges around the vertex in a planar embedding. This is not the case for the 1-string B_2-VPG-representations in our earlier construction [3]. One possible avenue towards showing that planar graphs do not always have an {L}-representation is to restrict the attention to ordered representations first. Thus, is there a planar graph that has no ordered {L}-representation?

References

[1] Andrei Asinowski, Elad Cohen, Martin Charles Golumbic, Vincent Limouzy, Marina Lipshteyn, and Michal Stern. String graphs of k-bend paths on a grid. *Electronic Notes in Discrete Mathematics*, 37:141–146, 2011.

[2] Andrei Asinowski, Elad Cohen, Martin Charles Golumbic, Vincent Limouzy, Marina Lipshteyn, and Michal Stern. Vertex intersection graphs of paths on a grid. *J. Graph Algorithms Appl.*, 16(2):129–150, 2012.

[3] Therese Biedl and Martin Derka. 1-string B_2-VPG representations of planar graphs. *Symposium on Computational Geometry (SoCG'15)*, 2015. To appear.

[4] Therese Biedl and Lesvia Elena Ruiz Velázquez. Drawing planar 3-trees with given face areas. *Comput. Geom.*, 46(3):276–285, 2013.

[5] Hans Bodlaender. Planar graphs with bounded treewidth. Technical Report RUU-CS-88-14, Rijksuniversiteit Utrecht, 1988.

[6] Jérémie Chalopin and Daniel Gonçalves. Every planar graph is the intersection graph of segments in the plane: extended abstract. In *ACM Symposium on Theory of Computing, STOC 2009*, pages 631–638. ACM, 2009.

[7] Jérémie Chalopin, Daniel Gonçalves, and Pascal Ochem. Planar graphs are in 1-string. In *Proceedings of the Eighteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA '07, pages 609–617, Philadelphia, PA, USA, 2007. Society for Industrial and Applied Mathematics.

[8] Jérémie Chalopin, Daniel Gonçalves, and Pascal Ochem. Planar graphs have 1-string representations. *Discrete & Computational Geometry*, 43(3):626–647, 2010.

[9] Steven Chaplick, Vít Jelínek, Jan Kratochvíl, and Tomás Vyskocil. Bend-bounded path intersection graphs: Sausages, noodles, and waffles on a grill. In *Graph-Theoretic Concepts in Computer Science - WG 2012*, volume 7551 of *Lecture Notes in Computer Science*, pages 274–285. Springer, 2012.

[10] Steven Chaplick and Torsten Ueckerdt. Planar graphs as VPG-graphs. *J. Graph Alg. Appl.*, 17(4):475–494, 2013.

[11] Hubert de Fraysseix, Patrice Ossona de Mendez, and János Pach. Representation of planar graphs by segments. *Intuitive Geometry*, 63:109–117, 1991.

[12] Gideon Ehrlich, Shimon Even, and Robert Endre Tarjan. Intersection graphs of curves in the plane. *J. Comb. Theory, Ser. B*, 21(1):8–20, 1976.

[13] Ehab S. El-Mallah and Charles J. Colbourn. Partitioning the edges of a planar graph into two partial k-trees. *Congr. Numerantium 66*, page 69–80, 1988.

[14] Stefan Felsner, Kolja B. Knauer, George B. Mertzios, and Torsten Ueckerdt. Intersection graphs of l-shapes and segments in the plane. In *Mathematical Foundations of Computer Science - MFCS 2014*, volume 8635 of *Lecture Notes in Computer Science*, pages 299–310. Springer, 2014.

[15] Mathew C. Francis and Abhiruk Lahiri. VPG and EPG bend-numbers of Halin graphs. *arXiv:1505.06036*, 2015.

[16] Rudof Halin. Studies on minimally n-connected graphs. In *Combinatorial Mathematics and its Applications*, pages 129–136. Academic Press, London, 1971.

[17] Irith Ben-Arroyo Hartman, Ilan Newman, and Ran Ziv. On grid intersection graphs. *Discrete Mathematics*, 87(1):41–52, 1991.

[18] Edward R. Scheinerman. *Intersection Classes and Multiple Intersection Parameters of Graphs.* PhD thesis, Princeton University, 1984.