Voronoi Games and Epsilon Nets

Aritra Banik[*] Jean-Lou De Carufel[†] Anil Maheshwari[†‡] Michiel Smid[†‡]

Abstract

Competitive facility location is concerned with the strategic placement of facilities by competing market players. In the *Discrete Voronoi Game* $VG(k, l)$, two players $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, strive to attract as many of n users as possible. Initially, $\mathcal{P}_1$ first chooses a set F of k locations in the plane to place its facilities. Then, $\mathcal{P}_2$ chooses a set S of l locations in the plane to place its facilities, where $S \cap F = \emptyset$. Finally, the users choose the facilities based on the nearest-neighbour rule. The goal for each player is to maximize the number of users served by its set of facilities.

By establishing a connection between $VG(2, 1)$ and ϵ-nets, we provide an algorithm running in $O(n \log^4 n)$ time to find a $\frac{7}{4}$-factor approximation of the optimal strategy of $\mathcal{P}_1$ in $VG(2, 1)$. We also prove that for any real number $0 < \alpha < 1$, there exists a placement of $\frac{42}{\alpha}$ facilities by $\mathcal{P}_1$ such that $\mathcal{P}_2$ can serve at most αn users by placing one facility.

1 Introduction

Competitive facility location is concerned with the strategic placement of facilities by competing market players. The main objective is to judiciously place a set of facilities, represented as points in the plane, serving a set of users such that certain optimality criteria are satisfied. Each facility has its *service zone*, consisting of the set of users it serves. Competitive facility location has been studied in several contexts [1, 7, 8, 9].

In this paper, we assume that the facilities are equally equipped in all respects, and a user always avails the service from its nearest facility. The *Voronoi Game* refers to the following facility location problem [1, 13]. Two players alternately place one point in the plane, until each of them has placed a given number of points. Then we subdivide the plane according to the nearest-neighbour rule. The player whose points control the larger area wins. To *solve* a Voronoi game corresponds to find an optimal strategy for each player.

In the *discrete* version [4], users are also represented as points in the plane (refer to Figure 1). In such a scenario, when the users choose the facilities based on the nearest-neighbour rule, the optimization criteria is to maximize the number of users served by each player. Moreover, in this paper, we study a version of the discrete Voronoi game where $\mathcal{P}_1$ first places all its facilities, after which $\mathcal{P}_2$ places all its facilities.

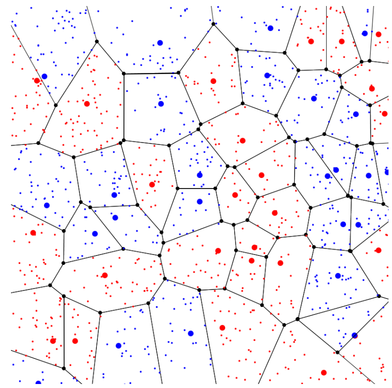

Figure 1: Distribution of users, denoted by the small points, among two competing market players denoted by red and blue large points.

Consider a set U of users and a set $\mathcal{F}$ of facilities. For every facility $f \in \mathcal{F}$, we define the *service zone* $U(f, \mathcal{F})$ as the set of users in U that are served by f. The *discrete Voronoi game* is a competitive facility location problem where given a set of users U, two competitive companies or players $\mathcal{P}_1$ and $\mathcal{P}_2$ place two disjoint sets of facilities. Any user $u_i \in U$ is said to be served by $\mathcal{P}_j$, $j \in \{1, 2\}$, if the facility closest to u_i is owned by $\mathcal{P}_j$. For any placements of facilities F and S by $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, the *payoff of* $\mathcal{P}_2$ (or the value of the game) $\mathcal{V}(F, S)$ is defined as the cardinality of the set of users in U that are served by a facility owned by $\mathcal{P}_2$. More formally, $\mathcal{V}(F, S) = |\bigcup_{f \in S} U(f, F \cup S)|$. Similarly, the payoff of $\mathcal{P}_1$ is $|U| - \mathcal{V}(F, S) = |\bigcup_{f \in F} U(f, F \cup S)|$. With these notations, the Discrete Voronoi Game on the plane, noted $VG(k, l)$, can be formally described as follows.

Definition 1 (Discrete Voronoi Game $VG(k, l)$) *Let U be a set of n users and $\mathcal{P}_1$ and $\mathcal{P}_2$ be two players. Initially, $\mathcal{P}_1$ chooses a set F of k locations in the plane to place its facilities. Then $\mathcal{P}_2$ chooses a set S of l locations in the plane to place its facilities, where $S \cap F = \emptyset$.*

1. *Given any choice of F by $\mathcal{P}_1$, the objective of $\mathcal{P}_2$ is to find a set S^* of l points that maximizes $\mathcal{V}(F, S)$,*

[*]Indian Statistical Institute
[†]Carleton University
[‡]Research supported by NSERC.

where the maximum is taken over all sets of l points $S \subset \mathbb{R}^2$. Formally, the objective of $\mathcal{P}_2$ is to find a set S^* of l points such that

$$\mathcal{V}(F, S^*) = \max_{\substack{S \subset \mathbb{R}^2 \\ |S|=l}} \mathcal{V}(F, S).$$

2. *The objective of $\mathcal{P}_1$ is to choose a set F^* of k facilities to minimize the maximum payoff of $\mathcal{P}_2$. In other words, the objective of $\mathcal{P}_1$ is to find a set F^* of k points such that*

$$\max_{\substack{S \subset \mathbb{R}^2 \\ |S|=l}} \mathcal{V}(F, S)$$

is minimized when $F = F^*$, where the minimum is taken of over all sets of k points $F \subset \mathbb{R}^2$.

Banik et al. [3] studied a version of $VG(k, k)$, where U is restricted to a line. They showed that, if the sorted order of points in U along the line is given, then for any given placement of facilities of $\mathcal{P}_1$, an optimal strategy for $\mathcal{P}_2$ can be computed in linear time. They also provided results for determining an optimal strategy for $\mathcal{P}_1$.

The discrete Voronoi game when the user set consists of a finite set of n points in $\mathbb{R}^2$ was first considered by Banik et al. [4], where they studied the following version of the game. The players $\mathcal{P}_1$ and $\mathcal{P}_2$ already own two sets of facilities F and S, respectively. The player $\mathcal{P}_1$ wants to place one more facility knowing that $\mathcal{P}_2$ will place another facility afterwards. This game is called the *One-Round Discrete Voronoi Game in Presence of Existing Facilities*, or One-Round-$VG(F, S)$ for short. The optimal strategy of $\mathcal{P}_2$, given any placement of $\mathcal{P}_1$, is identical to the solution of the *MaxCov* problem studied by Cabello et al. [5]. Consider a set U of users, two sets of facilities F and S, and any placement of a new facility f by $\mathcal{P}_1$. Let $U_1 \subseteq U$ denote the subset of users that are served by $\mathcal{P}_1$, in presence of F, S, and f. For every point $u \in U_1$, consider the *nearest facility disk* C_u centered at u and passing through the facility in $F \cup \{f\}$ which is closest to u. Note that a new facility s placed by $\mathcal{P}_2$ serves any user $u \in U_1$ if and only if $s \in C_u$. Let $\mathcal{C} = \{C_u | u \in U_1\}$. Any optimal strategy for $\mathcal{P}_2$ is a point which is inside a maximum number of circles among the circles in $\mathcal{C}$. This is the problem of finding the maximum depth in an arrangement of n disks, and can be computed in $O(n^2)$ time [2]. Banik et al. [4] study how this arrangement changes as f and s move in the plane. They provide a complete characterization of the event points and obtain an algorithm running in $O(n^8)$ time for computing an optimal placement of P_1.

In the One-Round-$VG(F, S)$, if $S = \emptyset$, the game resembles $VG(k, 1)$. The difference is that in $VG(k, 1)$, $\mathcal{P}_1$ can choose the location of all of its facilities whereas in the One-Round-$VG(F, \emptyset)$, $\mathcal{P}_1$ can only choose the location of one facility. Moreover, the solution to the One-Round-$VG(F, \emptyset)$ takes polynomial time, where the polynomial has a very high degree. In this paper, we focus on achieving approximate solutions to $VG(k, 1)$ with significantly better running times.

In Section 2, we provide an approximate solution to the optimal strategy of $\mathcal{P}_1$ in $VG(2, 1)$ by establishing a connection between $VG(2, 1)$ and ϵ-nets. To the best of our knowledge, this is the first time that Voronoi game is studied from the point of view of ϵ-nets. We also extend the study to provide a bound on the payoff of $\mathcal{P}_1$ in $VG(k, 1)$. We prove that for any real number $0 < \alpha < 1$, there exists a placement of $\frac{42}{\alpha}$ facilities by $\mathcal{P}_1$ such that $\mathcal{P}_2$ can serve at most αn users by placing one facility.

2 An Approximate Solution

In this section, we provide an approximate solution to $VG(2, 1)$ by establishing a connection between ϵ-nets and Voronoi games. Let us begin our discussion by defining what is an ϵ-net.

Definition 2 (Weak ϵ-net) *Consider any real number $\epsilon \in [0, 1]$. Let X be a finite set of points in $\mathbb{R}^2$ and $\mathcal{R}$ be a set of subsets of X. We call the pair $(X, \mathcal{R})$ a* range space. *The elements of X and $\mathcal{R}$ are called* points *and* ranges *of the range space, respectively. A finite set $N \in \mathbb{R}^2$ is a weak ϵ-net for $(X, \mathcal{R})$ if N intersects every set $K \in \mathcal{R}$ with $|K| > \epsilon |X|$.*

Mustafa and Ray [12] proved the following theorem.

Theorem 1 *Let P be a set of n points in $\mathbb{R}^2$. There exist two distinct points $z_1(P)$ and $z_2(P)$ such that any convex set containing at least $\frac{4}{7}n$ points of P also contains at least one point in $\{z_1(P), z_2(P)\}$.*

Given a set P of n points, Langerman et al. [10] described an $O(n \log^4 n)$ time algorithm to find $z_1(P)$ and $z_2(P)$. We will show if a set U of n users is given and if $\mathcal{P}_1$ places its facilities at points $z_1(U)$ and $z_2(U)$, $\mathcal{P}_1$ can guarantee a payoff of $\frac{3}{7}n$. Suppose $\mathcal{P}_1$ placed its facilities at points $z_1(U)$ and $z_2(U)$, and there exists a placement of facility s by $\mathcal{P}_2$ which serves more than $\frac{4}{7}n$ users. Observe that $\mathcal{P}_2$ will serve the set of users present in the Voronoi region of s in the Voronoi diagram of f_1, f_2 and s. The Voronoi region of s is a convex set which does not contain any of f_1 and f_2 but contains more than $\frac{4}{7}n$ points of U. This contradicts Theorem 1. Therefore we have the following observation.

Observation 1 *For any given set of n users U, there exist two points p_1 and p_2 such that placing its facilities at those two points, $\mathcal{P}_1$ can guarantee a payoff of at least $\frac{3}{7}n$.*

Next we prove that given any placement of facilities $\{f_1, f_2\}$ by $\mathcal{P}_1$, there exists a placement by $\mathcal{P}_2$ such that $\mathcal{P}_2$ can guarantee a payoff at least $\frac{n}{4}$. Let ℓ be the line joining f_1 and f_2 (see Figure 2). Denote the lines

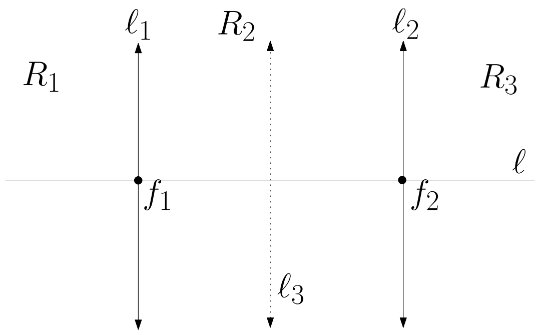

Figure 2: Minimum payoff of $\mathcal{P}_2$.

perpendicular to ℓ and passing through f_1 and f_2 by ℓ_1 and ℓ_2, respectively. Observe that ℓ_1 and ℓ_2 divides $\mathbb{R}^2$ into three regions R_1, R_2 and R_3, respectively. Observe that by placing one facility in R_1 or R_3, $\mathcal{P}_2$ can serve all the users in R_1 or R_3. Hence if one of R_1 or R_3 contains more than $\frac{n}{4}$ users, $\mathcal{P}_2$ can serve $\frac{n}{4}$ users. On the other hand if none of R_1 or R_3 contains more than $\frac{n}{4}$ users, R_2 contains at least $\frac{n}{2}$ users. Consider the line ℓ_3, parallel to ℓ_1 and ℓ_2 and bisecting the line segment joining f_1 and f_2. Observe that by placing a facility very close to f_1, on the line segment joining f_1 and f_2, $\mathcal{P}_2$ can serve all the users in the region bounded by the lines ℓ_1 and ℓ_3. Similarly by placing a facility very close to f_2, on the line segment joining f_1 and f_2, $\mathcal{P}_2$ can serve all the users in the region bounded by the lines ℓ_2 and ℓ_3. As R_2 contains at least $\frac{n}{2}$ users one of these two regions must contain at least $\frac{n}{4}$ users. Hence, $\mathcal{P}_2$ can still serve at least $\frac{n}{4}$ users. Therefore we have the following observation.

Observation 2 *Let a set U of n users and the placement of two facilities by $\mathcal{P}_1$ be given. There exists a placement of one facility by $\mathcal{P}_2$ such that $\mathcal{P}_2$ can guarantee a payoff of $\frac{n}{4}$.*

Denote the optimal payoff for $\mathcal{P}_1$ in $VG(2, 1)$ by λ. From Observation 2, we have $\lambda < \frac{3n}{4}$. From Observation 1, we know there exist a placement which guarantees a payoff of $\frac{3n}{7}$. Therefore, we have the following lemma.

Lemma 2 *There exists an algorithm running in $O(n \log^4 n)$ time to find a $\frac{7}{4}$-factor approximation of the optimal strategy of $\mathcal{P}_1$ in $VG(2, 1)$.*

We have already proved that in $VG(2, 1)$ there exists a placement strategy by $\mathcal{P}_1$ such that $\mathcal{P}_2$ can get at most $\frac{4}{7}n$ users. We want to extend this study for any real number $0 < \alpha < 1$. That is, given any $0 < \alpha < 1$, determine whether there exists an integer $f(\alpha)$ such that, in $VG(f(\alpha), 1)$, there is a placement strategy by $\mathcal{P}_1$ such that $\mathcal{P}_2$ can get at most αn users. It is known that for any convex range spaces and any $0 < \epsilon < 1$ there exists an ϵ-net of size $O(\frac{1}{\epsilon}\text{polylog}\frac{1}{\epsilon})$ [6]. Hence for any real number $0 < \alpha < 1$, $f(\alpha) \in O(\frac{1}{\alpha}\text{polylog}\frac{1}{\alpha})$. The question is whether $f(\alpha) \in O(\frac{1}{\alpha})$? We answer this question affirmatively.

Lemma 3 *Let be given a set U of n users, a set of facilities F placed by $\mathcal{P}_1$ and a facility s placed by $\mathcal{P}_2$, such that s serves αn users. There exists a circle which does not contain any of the facilities from F and contains at least $\lceil \frac{\alpha n}{6} \rceil$ users.*

Proof. Denote by U_s the set of at least αn users served by s. Consider any six rays emerging from s such that the angle between any two consecutive rays is $60°$ (see Figure 3). These six rays divide the plane into six regions. At least one of these regions contains at least $\frac{\alpha n}{6}$ users from U_s. Let λ be such a region and $U' \subset U_s$ be the users present in λ.

Consider any user $u \in U'$ which is farthest from s. Denote the circle centered at s and passing through u by C_s (see Figure 3).

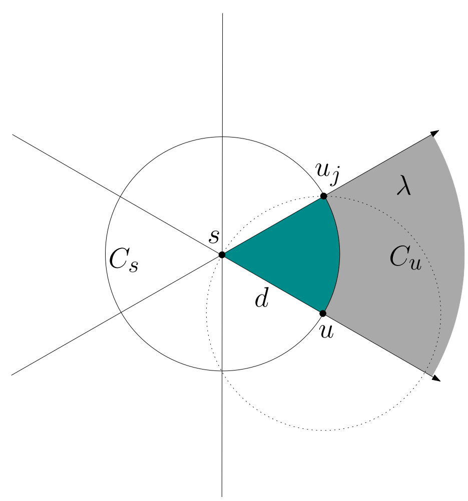

Figure 3: Illustration of lemma 3.

Let the distance between u and s be d. As $u \in U'$ is farthest from s, therefore all the users of λ lie in the region $\lambda \cap C_s$. Since the angle between the bounding lines of λ is $60°$, the maximum distance between any two points in $\lambda \cap C_s$ is d. Hence the circle C_u centered at u with radius d contains all the users in $\lambda \cap C_s$. Since u is served by s, C_u does not contain any other facility from F. Hence the result holds. □

The following theorem is due to Matoušek, Seidel and Welzl [11].

Theorem 4 *Let $0 < \epsilon \leq 1$ be a real number and let $\mathcal{D}$ be a family of disks. For every finite point set S in the plane there exists an ϵ-net with respect to $\mathcal{D}$ of size $O(1/\epsilon)$.*

We prove the following theorem using Lemma 3 and Theorem 4.

Theorem 5 *For any real number $0 < \alpha < 1$ and any set U of n users, there exists an integer $k \in O(\frac{1}{\alpha})$ such that in $VG(k, 1)$, $\mathcal{P}_1$ can choose k points to place its facilities such that $\mathcal{P}_2$ can serve at most αn users by placing one facility.*

Proof. Fix $\epsilon = \frac{\alpha}{6}$. Find the ϵ-net E for disks of size $k \in O(1/\epsilon)$ such that any disk which contains ϵn users contains at least one point from E. Therefore, $k \in O(1/\alpha)$. We claim that if $\mathcal{P}_1$ places its facilities at E, $\mathcal{P}_2$ will get at most αn users. Suppose there exists a placement of facility by $\mathcal{P}_2$ which serves $\alpha n + 1$ users. From Lemma 3, we know that there exist a disk which does not contain any point from E and contains $\lceil \frac{\alpha n+1}{6} \rceil > \epsilon n$ users. This contradicts the fact that E is an ϵ-net for the set of users. $\square$

Let P be any set of n points in the plane and $0 < \epsilon < 1$ be any real number. Given a set of points Q and a set of disks $\mathcal{D}$, we say that Q *pierces* $\mathcal{D}$ if for any disk $D \in \mathcal{D}$, $D \cap Q \neq \emptyset$. We know by Theorem 4 that for any $0 < \epsilon \leq 1$, there exists a set of $k \in O(1/\epsilon)$ points which pierces all the disks that contain ϵn points. However the constant hidden in the $O-$notation is fairly big. Next, we prove that given any set P of n points, there exists a set of $7/\epsilon$ points which pierces any disk that contains ϵn points.

Given a set of n points P, let the minimum disk that contains ϵn points from P be D^*. Consider the set $\mathcal{D}$ of all disks D such that $D^* \cap D \neq \emptyset$ and D contains at least ϵn points from P.

Lemma 6 *The set $\mathcal{D}$ can be pierced by 7 points.*

Proof. Let D^* be centered at c and denote the radius of D^* by r. Consider the disk D^{**} centered at c with radius $2r$ (see Figure 4). We construct a set Q, containing 7 points, that pierces $\mathcal{D}$. We first include c in Q. Consider any six rays emerging from c such that the angle between any two consecutive rays is 60° (see Figure 5). Let λ be any of the six sectors defined by these rays. Consider the set $\mathcal{D}_\lambda \subset \mathcal{D}$ of disks that do not contain c and whose centers are in λ. We show that there exists a point that pierces $\mathcal{D}_\lambda$.

Observe that the center of any disk in $\mathcal{D}_\lambda$ must lie outside of D^* because D^* is the minimum radius disk that contains ϵn points. The disks D^* and D^{**} intersect the boundary of λ in four points p_1, p_2, p_3 and p_4, respectively (see Figure 5). Using elementary geometry, we can show that there exists a point q such that

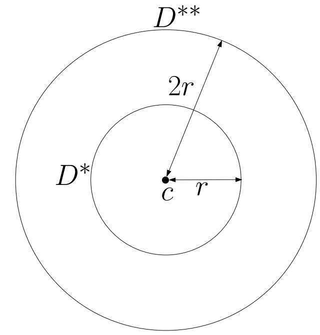

Figure 4: Minimum disk containing ϵn points.

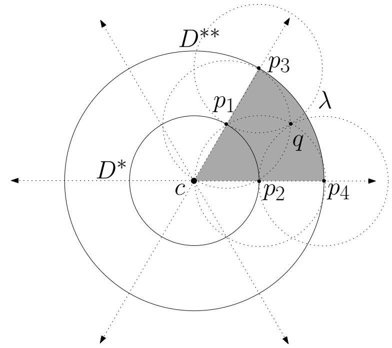

Figure 5: Illustration of Lemma 6.

$|p_1q| = |p_2q| = |p_3q| = |p_4q| = r$ (see Figure 5). Let $p \in \lambda \backslash D^*$ be the center of a disk D_i in $\mathcal{D}_\lambda$. We consider two cases: (1) $p \in \lambda \backslash D^{**}$ or (2) $p \in \lambda \cap D^{**}$.

1. Without loss of generality, assume that p is on the line joining p_2 and p_4 (see Figure 6). From the def-

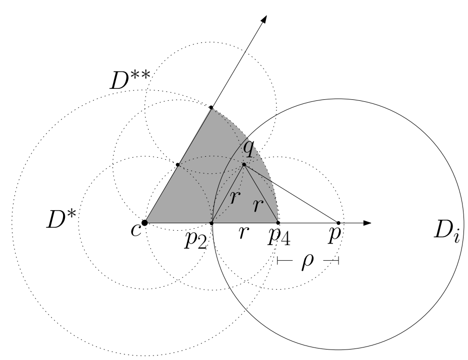

Figure 6: Illustration of the proof of Lemma 6, Case (1).

inition of q, we have $d(q, p_2) = d(q, p_4) = r$, where $d(a, b)$ denotes the distance between the points a and b. Furthermore, $d(p_2, p_4) = r$. Therefore, $\triangle qp_2p_4$ is an equilateral triangle. Hence, the angle $\angle qp_2p_4 = 60°$. Let the distance between p_4 and p be ρ. Let the radius of D_i be r_i. Since $D_i \cap D^* \neq \emptyset$, we have

$$r_i \geq r + \rho. \quad (1)$$

Consider the triangle $\triangle qp_4p$. From the triangle inequality we have

$$r + \rho \geq d(p, q). \quad (2)$$

From equation 1 and 2 we have $r_i \geq d(p, q)$. Hence, q pierces D_i.

2. Without loss of generality, assume that p is on the line joining p_2 and p_4 (see Figure 7). The radius

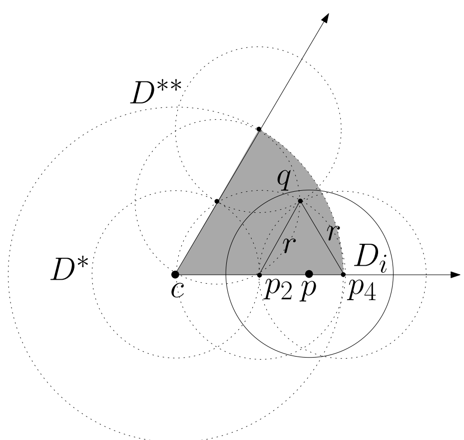

Figure 7: Illustration of the proof of Lemma 6, Case (2).

of D_i is at least r because D^* is the minimum-radius disk that contains ϵn points. As $\triangle qp_2p_4$ is equilateral, the distance from any point on the line p_2p_4 is less than $d(p_2, q) = r$. Hence, q pierces D_i.

□

Given a set P of n points, we provide an iterative algorithm to find an ϵ-net Q_ϵ of size $7/\epsilon$. At each stage of the algorithm, we find the minimum disk D^* that contains ϵn points of P. From Lemma 6, we know there exists a set Q of 7 points which pierces the set of all disks containing ϵn points of P and having a nonempty intersection with D^*. We include the points in Q to Q_ϵ and remove all points of P which are inside D^*. We continue this process until P contains no more than ϵn points. Therefore, the cardinality of Q_ϵ at the end of the process is at most $7/\epsilon$. We claim that this algorithm correctly finds an ϵ-net for P.

Suppose Q_ϵ is not an ϵ-net for P. Hence, there exists a disk $\hat{D}$, which contains at least ϵn points, that is not pierced by any of the points in Q_ϵ. Denote by D_i the minimum disk, which contains ϵn points from P, that we choose at stage i of the algorithm. If $\hat{D}$ did not intersect with any of the D_i's, then $\hat{D}$ would contain less than ϵn points. Therefore, let D_j be the first disk that has a nonempty intersection with $\hat{D}$. Notice that none of the points in $\hat{D}$ has been removed from P at earlier stages. Thus, from Lemma 6, $\hat{D}$ must be pierced by one of the 7 points chosen at stage j. Hence, we have the following theorem.

Theorem 7 *Given any set P of n points, there exists a set of $7/\epsilon$ points which pierces the set of all the disks that contain at least ϵn points.*

From Theorem 7 and Lemma 3, we have the following theorem.

Theorem 8 *For any real number $0 < \alpha < 1$, there exists a placement of $\frac{42}{\alpha}$ facilities by $\mathcal{P}_1$ such that $\mathcal{P}_2$ can serve at most αn users by placing one facility.*

3 Conclusion

Using an approach similar to that of Banik et al. [4], we can solve $VG(2, 1)$ exactly. If we fix the locations of the facilities (f_1 and f_2 for $\mathcal{P}_1$, and s for $\mathcal{P}_2$), we can define a *nearest facility disk* for each user (refer to Section 1 or to [4]). This defines an arrangement of cells. We claim that the boundary of each cell in this arrangement is made of circular arcs or line segments. By studying how this arrangement changes as we move f_1, f_2 and s, we can compute the optimal strategy for $\mathcal{P}_1$ and $\mathcal{P}_2$. However, this leads to a polynomial time algorithm, where the polynomial has a very high degree.

We would like to find a different approach to solve the problem exactly which would lead to a faster algorithm. This will be the subject of further research.

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