Expansive Motions for d-Dimensional Open Chains

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Abstract

We consider the problem of straightening chains in $d \geq 3$ dimensions, possibly embedded into higher dimensions, using expansive motions. For any $d \geq 3$, we show that there is an open chain in d dimensions that is not straight and not self-touching yet has no expansive motion. Furthermore, for any $\Delta > 0$ and $d \geq 3$, we show that there is an open chain in d dimensions that cannot be straightened using expansive motions when embedded into $\mathbb{R}^d \times [-\Delta, \Delta]$ (a bounded extra dimension). On the positive side, we prove that any open chain in $d \geq 2$ dimensions can be straightened using an expansive motion when embedded into $\mathbb{R}^{d+1}$ (a full extra dimension).

1 Introduction

Expansive motions have proved to be a powerful technique for reconfiguring planar linkages. The purpose of this paper is to determine how useful they can be in higher dimensions.

Expansive motions first proved useful by providing the key to solving the Carpenter's Rule Problem [7, 10], whether every planar chain linkage (forming a path or a cycle) could be universally reconfigured by motions that preserve edge lengths and avoid self-crossings. A positive answer was established by proving that every planar open chain that is not straight, and every planar closed chain that is not convex, has an *expansive* motion in the sense that no two vertices ever get closer together. While it is difficult to avoid self-crossings directly, expansiveness implies such avoidance, and expansive motions are easier to argue about because of their relation to tensegrities (explained below).

Expansive motions have since proved useful in establishing universal reconfigurability of other types of planar linkages. Streinu and Whiteley [11] extended the result to sufficiently short chains on a sphere, which has applications to rigid folding of single-vertex origami crease patterns. Connelly et al. [5] proved that chains of "slender adornments" can be universally configured, using expansive motions of an underlying chain linkage. This result was in turn useful for avoiding self-intersection in universal hinged dissections [1]. Expansive motions also led to the extensive study of "pointed pseudotriangulations", because pointed pseudotriangulations are the extreme rays in the cone of expansive motions [8, 10].

Beyond two dimensions, linkages have been studied, but not in the context of expansive motions. Chain linkages cannot be universally reconfigured in three dimensions [2], implying that some nonstraight open chain has no expansive motion. One might hope that the subset of 3D open chains that can be straightened can do so via expansive motions. In 4D and higher, the situation is more promising: chain linkages can be universally reconfigured [3], via fairly simple algorithms. A natural question, therefore, is whether all 4D and higher-dimensional chains have expansive motions.

Our results. Alas, we prove the existence of open chains in d dimensions, for all $d \geq 3$, that are not straight yet have no expansive motion. Furthermore, we can (for $d = 3$) guarantee that the chain can be straightened (by nonexpansive motions), shooting down the hope that such chains have expansive motions. We start by constructing a self-touching chain with this property, and then prove that there exist sufficiently small perturbations of the chain that still have the property and are non-self-touching.

Next we consider how many extra dimensions we have to add to d-dimensional space to guarantee expansive motions of a chain that lives in a d-dimensional (sub)space. On the one hand, we show that adding a bounded dimension $[-\Delta, \Delta]$ does not suffice: for any $\Delta > 0$, there is a d-dimensional open chain that has no expansive motion in $\mathbb{R}^d \times [-\Delta, \Delta]$. On the other hand, we show that a full extra dimension suffices: any d-dimensional open chain has an expansive motion all the way to straight in $d + 1$ dimensions.

2 Preliminaries

We begin by defining tensegrity frameworks, using a modified version of the notation in [4, 9].

An *abstract tensegrity* is an undirected graph G, with vertices $V(G)$ and edges $E(G) = B(G) \cup C(G) \cup S(G)$, where $B(G)$, $C(G)$, and $S(G)$ are pairwise disjoint. The edges $e \in B(G)$ are *bars* whose lengths are fixed. The edges $e \in C(G)$ are *cables* whose lengths cannot increase. The edges $e \in S(G)$ are *struts* whose lengths

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cannot decrease. A *tensegrity framework* $G(p)$ in d dimensions consists of an abstract tensegrity G and a mapping $p : V(G) \to \mathbb{R}^d$ assigning a location to each vertex in the abstract tensegrity. Equivalently, we may consider p to be a point in $\mathbb{R}^{nd}$, where $n = |V(G)|$. For convenience, we say that for each $k \in \{1, 2, \dots, d\}$, the function p_k gives the kth coordinate of p, so that $p(v_i) = (p_1(v_i), p_2(v_i), \dots, p_d(v_i))$ for each v_i.

A *linkage framework* $G(p)$ is a tensegrity framework such that $C(G) = S(G) = \emptyset$. An *open chain* of length n is a linkage framework $G(p)$ such that $V(G) = \{v_1, v_2, \dots, v_n\}$ and $B(G) = \{(v_1, v_2); (v_2, v_3); \dots ; (v_{n-1}, v_n)\}$. In this paper, we use the terms *chain* and *open chain* interchangeably.

We say that $G(q)$ is *another embedding* of a tensegrity framework $G(p)$ if $q(v_1) = p(v_1)$ and all of the following conditions hold:

$$\begin{aligned}
\forall(v_i, v_j) \in B(G) &: \|q(v_i) - q(v_j)\| = \|p(v_i) - p(v_j)\|, \\
\forall(v_i, v_j) \in C(G) &: \|q(v_i) - q(v_j)\| \leq \|p(v_i) - p(v_j)\|, \\
\forall(v_i, v_j) \in S(G) &: \|q(v_i) - q(v_j)\| \geq \|p(v_i) - p(v_j)\|.
\end{aligned}$$

Note that this relation is transitive, but not generally symmetric. Note also the non-standard requirement that $q(v_1) = p(v_1)$. This is used to ensure that any framework $G(p)$ where $(V(G), B(G) \cup C(G))$ is a connected graph has a bounded configuration space.

A tensegrity framework $G(p)$ is *self-touching* if there exist four distinct vertices v_i, v_j, v_k, and v_ℓ such that the edges $(v_i, v_j), (v_k, v_\ell) \in E(G)$, and the segment between $p(v_i)$ and $p(v_j)$ intersects with the segment between $p(v_k)$ and $p(v_\ell)$. If a tensegrity framework $G(p)$ with $B(G) \cup S(G) = V(G) \times V(G)$ is not self-touching, then any other embedding of $G(p)$ is also not self-touching.

A *motion* of a framework $G(p)$ is a continuous mapping from a time $t \in [0, 1]$ to a configuration p^t such that $p^0 = p$ and each $G(p^t)$ is another embedding of $G(p)$. A *rigid transformation* T is a distance-preserving transformation of $\mathbb{R}^d$. We use the notation $T(p)$ to denote the result of applying T to every vertex location $p(v_i)$. A motion is rigid if at every time t, there exists a rigid transformation T such that $p^t = T(p)$. A framework $G(p)$ is *rigid* if all motions of $G(p)$ are rigid motions.

An expansive motion is a motion where the distance between any pair of vertices is always non-decreasing. More formally, an *expansive motion* is a motion p^t such that for all pairs of vertices v_i, v_j, and for all times $t < t'$, $\|p^t(v_i) - p^t(v_j)\| \leq \|p^{t'}(v_i) - p^{t'}(v_j)\|$. We say that a tensegrity framework $G(p)$ is *rigid under expansive motion* if any expansive motion p^t is rigid.

An alternate embedding $G(q)$ is *reachable* from $G(p)$ if there is some motion p^t of $G(p)$ such that $p^0 = p$ and $p^1 = q$. A framework $G(p)$ is *locked* if there exists an alternate embedding of $G(p)$ that is not reachable from $G(p)$. A chain $G(p)$ is *straight* if it is not self-touching and all vertices lie along a line. Note that all straight chains are rigid under expansive motions.

An *infinitesimal motion* of a framework $G(p)$ assigns a velocity vector $u(v_i)$ to every vertex v_i which satisfies the following constraints:

$$\begin{aligned}
\forall(v_i, v_j) \in B(G) &: (u(v_i) - u(v_j)) \cdot (p(v_i) - p(v_j)) = 0, \\
\forall(v_i, v_j) \in C(G) &: (u(v_i) - u(v_j)) \cdot (p(v_i) - p(v_j)) \leq 0, \\
\forall(v_i, v_j) \in S(G) &: (u(v_i) - u(v_j)) \cdot (p(v_i) - p(v_j)) \geq 0.
\end{aligned}$$

These equations can be derived by taking the first derivative of the equations used to test whether some p^t is a motion for the framework $G(p)$, and setting $t = 0$. An infinitesimal motion of $G(p)$ is a rigid infinitesimal motion if it is equal to the derivative of some rigid motion at time $t = 0$. We say that $G(p)$ is infinitesimally rigid if all infinitesimal motions of $G(p)$ are rigid.

For a fixed choice of p, each of the infinitesimal motion equations is a linear inequality over u. Hence, if $G(p)$ is a linkage framework, then the constraints for infinitesimal motions can be written as a matrix with $|E(G)|$ rows and dn columns. The rank of this matrix can be used to calculate the number of degrees of freedom for infinitesimal motions. Hence, a linkage framework is infinitesimally rigid if and only if the rank of the matrix is equal to $dn - d(d + 1)/2$. This gives a simple way to test for infinitesimal rigidity, but only for linkages.

One way to test whether a tensegrity framework is infinitesimally rigid involves a concept called stress. A *stress* on a tensegrity framework $G(p)$ assigns a scalar value $s(v_i, v_j)$ to each edge $(v_i, v_j) \in E(G)$. Intuitively, the stress on some edge (v_i, v_j) applies a force proportional to $s(v_i, v_j)$ to both v_i and v_j. A stress $s(v_i, v_j)$ is known as an *equilibrium stress* if it satisfies the following constraint:

$$\forall v_i \in V(G), k \in \{1, \dots, d\} : \sum_{v_j : (v_i, v_j) \in E(G)} s(v_i, v_j) \cdot (p_k(v_i) - p_k(v_j)) = 0.$$

Roth and Whiteley [9] showed the following:

Theorem 1 [9] *Let $G(p)$ be a tensegrity framework. Let $\overline{G}$ be an abstract tensegrity framework such that $V(\overline{G}) = V(G)$, $B(\overline{G}) = E(G)$, and $C(\overline{G}) = S(\overline{G}) = \emptyset$. Then $G(p)$ is infinitesimally rigid if and only if $\overline{G}(p)$ is infinitesimally rigid and there exists an equilibrium stress on $G(p)$ such that $s(v_i, v_j) > 0$ for all cables (v_i, v_j) and $s(v_i, v_j) < 0$ for all struts (v_i, v_j).*

This theorem makes it easier to show the infinitesimal rigidity of a tensegrity.

One key theorem that we use several times in this paper is a result by Connelly [4] which has come to be known as sloppy rigidity [6]. Intuitively, if a framework $G(p)$ is rigid, then even if the edge lengths vary by

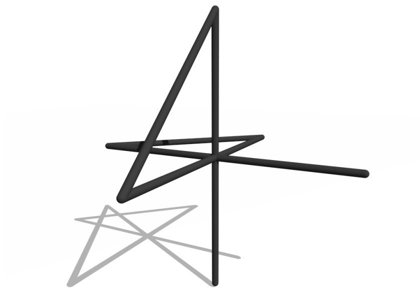

Figure 1: The three-dimensional version of the self-touching chain used in Lemma 3.

some small δ, any motion can only perturb the vertices of $G(p)$ by some small amount. Hence, $G(p)$ remains locked even with weaker constraints on the edge lengths. The theorem uses the idea of a rigidity neighborhood. A *rigidity neighborhood* U_p of some rigid framework $G(p)$ is any set that contains p and all of its rigid transformations, but does not contain any other q such that $G(q)$ is an alternate embedding of $G(p)$.

Theorem 2 [4] *Let $G(p)$ be rigid in $\mathbb{R}^n$, and let U_p be a rigidity neighborhood of p for $G(p)$. Let $\varepsilon > 0$ be given. Then there is some $\delta > 0$ such that, if $q \in U_p$ and the following conditions hold*

$$\begin{aligned} &\forall (v_i, v_j) \in C(G) \cup B(G) : \\ &\qquad \|q(v_i) - q(v_j)\|^2 < \|p(v_i) - p(v_j)\|^2 + \delta, \\ &\forall (v_i, v_j) \in S(G) \cup B(G) : \\ &\qquad \|q(v_i) - q(v_j)\|^2 > \|p(v_i) - p(v_j)\|^2 - \delta, \end{aligned}$$

then there is a rigid transformation T of $\mathbb{R}^d$ such that $\|T(q) - p\| < \varepsilon$.

3 Chains Rigid Under Expansive Motions

In this section, we show that for any $d \geq 3$, there exists a non-straight chain in d dimensions that is rigid under expansive motions. To do so, we begin by giving a self-touching chain that is rigid under expansive motions, which we later modify to make non-self-touching.

3.1 Self-Touching

Lemma 3 *For any $d \geq 3$, there exists a self-touching configuration of a chain in d dimensions that is rigid under expansive motions but is not straight.*

Proof. Consider the chain $G(p)$ of length $2d$ with

$$p_k(v_i) = \begin{cases} 1 & \text{if } i = 2k-1, \\ -1 & \text{if } i = 2k, \\ 0 & \text{otherwise,} \end{cases}$$

where $1 \leq k \leq d$ and $1 \leq i \leq 2d$. Intuitively, this chain is the result of connecting d bars, each of length 2, and each of which lies along one of the axes of $\mathbb{R}^d$ and is centered on the origin. The three-dimensional version of this chain is depicted in Figure 1.

Consider the tensegrity framework $G'(p)$ obtained by adding a strut between every pair of unconnected vertices. Proving that $G'(p)$ is rigid is equivalent to proving that $G(p)$ is rigid under expansive motions. To show this, we must first provide an equilibrium stress for the tensegrity that is negative on every strut. We define the stress $s(v_i, v_j)$ between two vertices $v_i \neq v_j$ as follows:

$$s(v_i, v_j) = \begin{cases} d-1 & \text{if } \exists k \text{ s.t. } \{i,j\} = \{2k-1, 2k\}, \\ -1 & \text{otherwise.} \end{cases}$$

It is easy to verify that this is an equilibrium stress of $G'(p)$, and that it is negative on all of the struts of $G'(p)$.

To show that $G'(p)$ is infinitesimally rigid, we must also show that replacing every strut in the tensegrity with a bar results in a linkage that is infinitesimally rigid. Consider the linkage framework $\overline{G}(p)$ that results from this process. There is a bar between every pair of vertices, so the linkage is clearly rigid. To see that it is infinitesimally rigid, we use an inductive argument on the number of dimensions for the chain.

We claim that, for any $d \geq 3$, there exists a set of $d(d+1)/2$ linear equations that, when combined with the constraints on the velocities for an infinitesimal motion, yield only the zero solution. This claim shows that the framework has $d(d+1)/2$ degrees of freedom, and is therefore infinitesimally rigid.

We show this claim by induction on d. The claim can be verified for $d = 3$. Assume by induction that there exist $d(d+1)/2$ linear equations that, when combined with the $2d^2 - d$ constraints for the d-dimensional version of this chain, restrict the space of solutions so that no infinitesimal motions are allowed. Now consider the $(d+1)$-dimensional version of this chain. Any infinitesimal motion must satisfy $2(d+1)^2 - (d+1)$ linear constraints, one for each edge. The first $2d$ vertices of the chain have the same coordinates as the vertices of the d-dimensional version of the chain. Hence, the $2d^2 - d$ constraints corresponding to each edge among those vertices are the same for the two chains. By adding the $d(d+1)/2$ linear equations guaranteed to exist by the inductive assumption, we ensure that, for any $k \neq d+1$ and any $i \notin \{2d+1, 2d+2\}$, $u_k(v_i) = 0$.

We now add $(d+1)$ more linear equations setting $u_k(v_{2d+1}) = 0$ for all $k \in \{1, \ldots, d+1\}$, for a total of $(d+1)(d+2)/2$ extra equations, which is precisely what we wanted. Consider the possible values for $u_{d+1}(v_i)$, for some $i \notin \{2d+1, 2d+2\}$. The edge between v_{2k-1} and v_{2d+1} results in the following equation:

$$(u(v_{2k-1}) - u(v_{2d+1})) \cdot (p(v_{2k-1}) - p(v_{2d+1})) = 0.$$

The vector $(p(v_{2k-1}) - p(v_{2d+1}))$ is non-zero in two coordinates: k and $d+1$. In addition, the $d+1$ equations we added ensure that $u(v_{2d+1}) = (0, 0, \ldots, 0)$. Hence, the above equation becomes

$$1 \cdot u_k(v_{2k-1}) + -1 \cdot u_{d+1}(v_{2k-1}) = 0.$$

The $d(d+1)/2$ equations from the inductive step ensure that $u_k(v_{2k-1}) = 0$. Hence, $u_{d+1}(v_{2k-1}) = 0$. A similar argument shows that $u_{d+1}(v_{2k}) = 0$ for any $k \neq d+1$. This means that the equations we have selected ensure that the velocity of every vertex $v_i \neq v_{2d+2}$ is zero.

Now consider the velocity of v_{2d+2}. The edge between v_{2d+1} and v_{2d+2} gives us the equation:

$$2 \cdot u_{d+1}(v_{2d+1}) - 2 \cdot u_{d+1}(v_{2d+2}) = 0.$$

Hence, we know that $u_{d+1}(v_{2d+2}) = 0$. Now consider the edge between v_{2d+2} and v_{2k-1}, which results in the following equation:

$$(u(v_{2k-1}) - u(v_{2d+2})) \cdot (p(v_{2k-1}) - p(v_{2d+2})) = 0.$$

The velocity $u(v_{2k-1})$ is zero in all coordinates, and $(p(v_{2k-1}) - p(v_{2d+2}))$ is nonzero only in coordinates k and $d+1$. Therefore, the equation becomes:

$$1 \cdot -u_k(v_{2d+2}) + 1 \cdot -u_{d+1}(v_{2d+2}) = 0.$$

We have shown that $u_{d+1}(v_{2d+2}) = 0$, and so it must be that $u_k(v_{2d+2}) = 0$. $\square$

3.2 Non-Self-Touching

Now that we have shown the existence of a self-touching chain that is rigid under expansive motions, we use the sloppy rigidity results from Theorem 2 to show that there is also a non-self-touching chain that is rigid under expansive motions.

Theorem 4 *For any $d \geq 3$, there exists a non-self-touching configuration of a chain in d dimensions that is rigid under expansive motions but is not straight.*

Proof. Consider the chain $G(p)$ specified in Lemma 3. Add a strut between every pair of vertices that is not already connected by a bar to obtain a rigid tensegrity framework $G'(p)$. Because $G'(p)$ is rigid, there must exist some c such that any other embedding $G'(q)$ that is not a rigid transformation of p has $\|p - q\| > c$.

Now let U_p be a neighborhood of p such that $\forall r \in U_p$, there exists a rigid transformation T such that $\|T(r) - p\| < c$. By definition, U_p is a rigidity neighborhood of p. Let $\varepsilon = c/2$, be the value we use for Theorem 2, and let $\delta > 0$ be the resulting sloppiness.

If $\delta > \varepsilon/n$, set $\delta = \varepsilon/n$. Randomly perturb each vertex in p by a distance less than $\delta/2$ to get a framework $G'(p^*)$. Then any alternate embedding of $G'(p^*)$ has the property that its edge lengths violate the length constraints for an alternate embedding of $G'(p)$ by a distance of at most δ. By Theorem 2, if there is any $q \in U_p$ such that $G'(q)$ is another embedding of $G'(p^*)$, then there must exist a rigid transformation T such that $\|T(q) - p\| < \varepsilon$, and therefore

$$\|T(q) - p^*\| < \|T(q) - p\| + \|p - p^*\| < \varepsilon + n\delta/2 \leq \frac{3c}{4}.$$

Hence, any motion of $G'(p^*)$ cannot result in any framework $G'(q)$ such that $q \notin U_p$. As a result, $G'(p^*)$ is locked, and all reachable alternative embeddings of $G'(p^*)$ are not straight.

Define the function $f(q)$ to be the sum of all pairwise distances between the points in $G'(q)$. The set of alternate embeddings reachable from $G'(p^*)$ is both closed and bounded, so we can pick an embedding $G'(q^*)$ from that set that maximizes $f(\cdot)$.

Consider any alternate embedding $G(r)$ reachable from $G(q^*)$. Because $G'(r)$ is an alternate embedding of $G'(q^*)$, we know that for each edge $(v_i, v_j) \in E(G')$, $\|r(v_i) - r(v_j)\| \geq \|q^*(v_i) - q^*(v_j)\|$. Hence, we have

$$\sum_{i,j} \|r(v_i) - r(v_j)\| \geq \sum_{i,j} \|q^*(v_i) - q^*(v_j)\|,$$
$$f(r) \geq f(q^*).$$

By transitivity, $G'(r)$ is also a reachable alternate embedding of $G'(p^*)$. By definition of q^*, this means that $f(r) \leq f(q^*)$, and therefore that $f(r) = f(q^*)$. As a result, it must be that $\|r(v_i) - r(v_j)\| = \|q^*(v_i) - q^*(v_j)\|$ for all edges (v_i, v_j). Then r is a rigid transformation of q^*. So $G(q^*)$ is rigid under expansive motions.

Because of the random perturbations used to construct p^*, and because $d \geq 3$, no four vertices can lie in the same plane. Hence, $G'(p^*)$ must be non-self-touching. Because any alternate embedding of $G'(p^*)$ cannot decrease the distance between any two vertices, all alternate embeddings are also non-self-intersecting. This means that $G(q^*)$ is also not self-touching. $\square$

In Theorem 4, we give a non-constructive proof of the existence of a non-self-touching chain that is rigid under expansive motions in $d \geq 3$ dimensions. The following conjecture would provide a way to construct such a chain, but has only been verified for $3 \leq d \leq 8$.

Conjecture 1 *In $d \geq 3$ dimensions, the non-self-touching chain given by the coordinates*

$$p_k(v_i) = \begin{cases} 1 & \text{if } i = 2k-1, \\ -1 & \text{if } i = 2k, \\ 0.01 & \text{if } \lceil i/2 \rceil + 1 \equiv k \pmod{d}, \\ 0 & \text{otherwise}, \end{cases}$$

where $1 \leq k \leq d$ and $1 \leq n \leq 2d$, is rigid under expansive motions.

For $d = 3$, we can additionally prove that the constructed chain can be straightened: either end link can be folded by itself to extend the next link, thus effectively reducing the number of links to 4, which implies that the chain can be straightened [2].

3.3 Bounded Extra Dimension

We have shown that for any $d \geq 3$, there exists a d-dimensional chain that is rigid under expansive motions, but not straight. This naturally raises the question of how much space is required to expansively straighten a d-dimensional chain. In this section, we show that for any Δ, there exists a d-dimensional chain that cannot be straightened using an expansive motion in $\mathbb{R}^d \times [-\Delta, \Delta]$. We begin by proving the following lemma.

Lemma 5 *For any non-straight chain $G(p)$ in d dimensions that is rigid under expansive motions, there exists a constant $\Delta > 0$ such that $G(p)$ cannot be straightened using an expansive motion in $\mathbb{R}^d \times [-\Delta, \Delta]$.*

Proof. Let $G(p)$ be a non-straight chain in d dimensions that is rigid under expansive motions. Add a strut between every pair of vertices that are not connected by a bar, and call the resulting framework $G'(p)$. Then our goal is to show that $G'(p)$ is locked in $\mathbb{R}^d \times [-\Delta, \Delta]$.

Let $G'(q)$ be any alternate embedding of $G'(p)$ in $\mathbb{R}^d \times [-\Delta, \Delta]$. Consider projecting $G'(q)$ into d dimensions by omitting the last coordinate to get a framework $G'(r)$. For any v_i and v_j,

$$\begin{aligned}&\|q(v_i) - q(v_j)\|^2 = \\ &\qquad \|r(v_i) - r(v_j)\|^2 + (q_{d+1}(v_i) - q_{d+1}(v_j))^2.\end{aligned}$$

Because $G'(q)$ is embedded in $\mathbb{R}^d \times [-\Delta, \Delta]$, we know that $0 \leq (q_{d+1}(v_i) - q_{d+1}(v_j))^2 \leq 4\Delta^2$. Hence we have

$$\begin{aligned}&\|r(v_i) - r(v_j)\|^2 \geq \|q(v_i) - q(v_j)\|^2 - 4\Delta^2, \text{ and} \\ &\|r(v_i) - r(v_j)\|^2 \leq \|q(v_i) - q(v_j)\|^2.\end{aligned}$$

This means that any alternate embedding of $G'(p)$ in $\mathbb{R}^d \times [-\Delta, \Delta]$ corresponds to an alternate embedding of $G'(p)$ in $\mathbb{R}^d$ where the edge lengths are allowed to vary by at most $4\Delta^2$. Hence, if $G'(p)$ is locked in $\mathbb{R}^d$, even when the edge lengths are allowed to vary by up to $4\Delta^2$, then $G'(p)$ is locked in $\mathbb{R}^d \times [-\Delta, \Delta]$.

Because $G'(p)$ is rigid, we know that there is some constant c such that any other embedding $G'(q)$ that is not a rigid transformation of p has $\|p - q\| > c$. Let U_p be a neighborhood of p such that $\forall r \in U_p$, there is a rigid transformation T such that $\|T(r) - p\| < c$. Then U_p is a rigidity neighborhood of p. Let $\varepsilon < c$ be the value we use for Theorem 2, and let $\delta > 0$ be the resulting sloppiness.

Consider any $r \in U_p$ satisfying these constraints:

$$\begin{aligned}&\forall (v_i, v_i) \in B(G') \cup S(G') : \\ &\qquad \|r(v_i) - r(v_j)\|^2 \geq \|p(v_i) - p(v_j)\|^2 - 4\Delta^2, \\ &\forall (v_i, v_i) \in B(G') \cup C(G') : \\ &\qquad \|r(v_i) - r(v_j)\|^2 \leq \|p(v_i) - p(v_j)\|^2.\end{aligned}$$

Then if $4\Delta^2 \leq \delta$, there is a rigid transformation T such that $\|T(r) - p\| < \varepsilon$. This means that if $4\Delta^2 \leq \delta$, then $G'(p)$ is locked in $\mathbb{R}^d \times [-\Delta, \Delta]$. □

Theorem 6 *For any $d \geq 3$ and any $\Delta > 0$, there is a non-self-touching chain in d dimensions that cannot be straightened using an expansive motion in $\mathbb{R}^d \times [-\Delta, \Delta]$.*

Proof. By Theorem 4, there is a non-self-touching chain $G(p^*)$ in d dimensions that is rigid under expansive motions. By Lemma 5, there exists some Δ^* such that $G(p^*)$ cannot be expansively straightened in $\mathbb{R}^d \times [-\Delta^*, \Delta^*]$. Multiply the coordinates of p^* by Δ/Δ^* to get a new framework $G(p)$. For the sake of contradiction, say that $G(p)$ can be straightened in $\mathbb{R}^d \times [-\Delta, \Delta]$ using some expansive motion q^t. Multiply all coordinates of q^t by a factor of Δ^*/Δ to get a motion r^t in $\mathbb{R}^d \times [-\Delta^*, \Delta^*]$. Because q^t is a motion of p, the scaled r^t is a motion of p^*. Hence, if $G(p)$ can be straightened in $\mathbb{R}^d \times [-\Delta, \Delta]$, then $G(p^*)$ can be straightened in $\mathbb{R}^d \times [-\Delta^*, \Delta^*]$, which results in a contradiction. □

4 Expansive Motions in Higher Dimensions

We have shown that for any Δ, there is a d-dimensional chain that cannot be expansively straightened in $\mathbb{R}^d \times [-\Delta, \Delta]$. In this section, we show that any d-dimensional chain can be expansively straightened in $\mathbb{R}^{d+1}$, thus resolving the question of how much space is required to straighten expansively.

Theorem 7 *Any chain $G(p)$ in d dimensions can be straightened using an expansive motion when embedded in $(d + 1)$-dimensional space.*

Proof. For each v_i, we define z_i to be the length along the chain between v_1 and v_i. More formally, we let

$$z_i = \sum_{j=1}^{i-1} \|p(v_j) - p(v_{j+1})\|.$$

Note that for any i and j, $|z_i - z_j| \geq \|p(v_i) - p(v_j)\|$. The motion q used to straighten the chain will be:

$$q_k^t(v_i) = \begin{cases} z_i \cdot \sin\left(\frac{\pi t}{2}\right) & \text{if } k = d + 1, \\ p_k(v_i) \cdot \cos\left(\frac{\pi t}{2}\right) & \text{otherwise.} \end{cases}$$

At time $t = 0$, the location of vertex v_i will be $(p_1(v_i), \ldots, p_d(v_i), 0)$. At time $t = 1$, the location of vertex v_i will be $(0, 0, \ldots, 0, z_i)$. So our motion will cause all of the points to form a line. Consider how the square of the distance between v_i and v_j will change over time. A short derivation shows that

$$\frac{\|q^t(v_i) - q^t(v_j)\|^2}{\|p(v_i) - p(v_j)\|^2} = \left(\frac{(z_i - z_j)^2}{\|p(v_i) - p(v_j)\|^2} - 1\right) \sin^2\left(\tfrac{\pi t}{2}\right) + 1.$$

For adjacent v_i, v_j, we know that $\|p(v_i) - p(v_j)\| = z_i - z_j$, and therefore the length of the edge is preserved. But is the motion expansive over $t \in [0, 1]$? We know that $\sin^2\left(\frac{\pi t}{2}\right)$ is a non-decreasing function over the interval $[0, 1]$. We also know that $(z_i - z_j)^2 \geq \|p(v_i) - p(v_j)\|^2$. Hence, the distance between v_i and v_j is non-decreasing, meaning that the motion is expansive. □

Lemma 8 *For any $\Delta > 0$, any three-dimensional open chain can be straightened in $\mathbb{R}^3 \times [-\Delta, \Delta]$.*

Proof. Let $G(p)$ be a three-dimensional open chain that is not straight. To straighten $G(p)$, we first use a modified version of the motion from Theorem 7. Define z_i as we did in Theorem 7. Let $\theta = \arcsin(\Delta/z_n)$. The motion we use is

$$q_k^t(v_i) = \begin{cases} z_i \cdot \sin(\theta t) & \text{if } k = 4, \\ p_k(v_i) \cdot \cos(\theta t) & \text{otherwise.} \end{cases}$$

A similar analysis to Theorem 7 shows that this motion preserves the lengths of all bars, and does not cause the chain to become self-intersecting. In addition, we have selected the motion function so that $0 \leq p_{d+1}(v_i) \leq \Delta$.

The result of this motion is an alternate embedding $G(q)$ with the following coordinates:

$$q_k(v_i) = \begin{cases} (z_i \cdot \Delta)/z_n & \text{if } k = 4, \\ p_k(v_i) \cdot \sqrt{1 - (\Delta/z_n)^2} & \text{otherwise.} \end{cases}$$

To straighten this new chain, we hold $v_2, \ldots, v_n$ fixed in place, and swing v_1 around so that it is collinear with v_2 and v_3. More specifically, consider the set of possible locations for v_1 that preserve the fourth coordinate of v_1 and the length of the rigid bar between v_1 and v_2. This corresponds to a sphere. The initial location of v_1 is one point on the sphere; the location that will straighten the bars (v_1, v_2) and (v_2, v_3) is another point on the sphere. We can use a motion along the shortest path to get from one to the other. Once the two bars (v_1, v_2) and (v_2, v_3) have been straightened, we can treat them as a single bar and repeat until the whole chain is straightened. □

5 Open Problem

The main question left open by this work is whether every *closed* chain initially in d dimensions has an expansive motion in $d+1$ dimensions to a (planar) convex configuration. Such a result would be a natural extension to our positive result for open chains. (Our negative results extend to closed chains, simply by doubling our open chains into an Euler tour.)

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