Isotopic Fréchet Distance

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Abstract

We present a variant of the Fréchet distance (as well as geodesic and homotopic Fréchet distance) which forces the motion between the input objects to follow an ambient isotopy. This provides a measure of how much you need to continuously deform one shape into another while maintaining topologically equivalently shapes throughout the deformation.

1 Introduction

We are interested in defining a distance measure between two (homeomorphic) shapes. This measure has a number of potential applications in computer graphics and vision, such as assessing the error when approximating a continuous function by a discrete one, or evaluating the similarity between two shapes. We propose a new distance measure which intuitively is the least effort of morphing a source shape into a target shape, such that each intermediate shape during the morph is homeomorphic to the source. For a given morph, this "effort" is measured as the maximum distance traveled by any point on the source shape.

Our measure is closely related to Fréchet distance, which can be defined as the least travel distance among all possible deformations between the two shapes. Homotopic Fréchet distance [4] further restricts the deformations to be continuous, particularly in the presence of obstacles. However, the intermediate shapes during the deformation may not be homeomorphic to the source shape. For example, they may have self-intersections even though the source shape is intersection-free. Our measure, called *isotopic Fréchet Distance*, enforces the deformation of the source shape to induce a continuous deformation of the ambient space.

Two other distance measures similar to ours, in the special case of curves, were geodesic width [6] and minimum deformation area [10]. Geodesic width considers a class of deformations between two planar curves that is more restricted than what we consider in this work, in that no two intermediate curves during the deformation can intersect. Note that this restriction means that geodesic width is applicable only to non-intersecting curves. The deformation area is defined between two curves lying on any 2-manifold, and considers a similar class of deformations as in our work. The key difference is that the deformation area evaluates the "effort" of morphing as the area swept by the deformation, while the isotropic Fréchet Distance considers the longest distance traveled. Practical work on this problem has also been done, although with no real guarantee of optimality [9].

In this paper, we formulate isotropic Fréchet Distance and compare it with homotopic Fréchet distance. In particular, we give an example in 2D where this new measure better characterizes the dissimilarity between two curves. We also briefly touch on the challenges in computing the measure and its potential applications.

2 Definitions

Consider two homeomorphic subsets A and B of a metric space M. Often M will be Euclidean space or Euclidean space with obstacles removed from it. There are a variety of ways to measure how "close" A and B are. These include Hausdorff distance, Fréchet distance, geodesic Fréchet distance and homotopic Fréchet distance. Hausdorff distance measures how large of a neighborhood of A is needed to contain B and vice-versa.

Definition 1 *Given $A, B \subset M$, the* Hausdorff distance *between them is*

$$\mathcal{H}(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$$

However, Hausdorff distance is solely based on geometry and ignores the topology of both A and B. Fréchet distance considers all possible homeomorphic pairings between points in A and B and how far away paired points are. The distance is defined to be the minimum over all homeomorphisms between A and B of the maximum distance between any pair identified points. In many applications, Fréchet distance is only defined for curves; this definition generalizes this for arbitrary A and B, see [3] for a similar definition. Unless otherwise specified, all maps are assumed to be continuous.

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Definition 2 *Given $A, B \subset M$ with $X \cong A \cong B$, the* Fréchet distance *between them is*

$$\mathcal{F}(A,B) = \inf_{\substack{f,g : X \to M \\ f(X)=A,\, g(X)=B}} \sup_{x \in X} d(f(x), g(x))$$

Following geodesic paths between identified pairs of points (and around any obstacles) gives a way to deform one shape to another, yielding the geodesic Fréchet distance [5]. However, under either geodesic or standard Fréchet distance, nearby points do not follow similar paths when obstacles are present in the underlying space. For example, if A and B are curves and the geodesics are thought of as the traditional "dog leash" connecting the curves, the leash might jump discontinuously over any obstacles in the space.

Homotopic Fréchet distance [4] restricts Fréchet distance still further and only considers continuous deformations of one shape to another. For these definitions we must assume that M has a Riemannian metric or some other structure that allows the measurement of the length of curves.

Note that in the following definition, we consider X to be an abstract representation of A and B (which are homeomorphic), allowing us to match different pairs of points in A and B by fitting them to points in the reference shape X. In previous work, this distance was only defined if A and B were curves, and it was assumed that the parametrizations of the curves were non-decreasing. However, if only monotonic parametrizations are considered then the infimum does not change. Notice that monotonic parametrizations are homeomorphisms, so the following definition does generalize the definition of homotopic Fréchet distance to general spaces.

Definition 3 *Given $A, B \subset M$ with $X \cong A \cong B$, the* homotopic Fréchet distance *between them is*

$$\overline{\mathcal{F}}(A,B) = \inf_{\substack{h : X \times [0,1] \to M \\ h(X,0)=A,\, h(X,1)=B}} \max_{x \in X} \operatorname{len} h(x,\cdot)$$

Continuous deformations of one space to another can change the topology along the way. If we want to ensure that all intermediate spaces are identical and are embedded into Euclidean space then we replace homotopies by ambient isotopies. In essence, isotopic Fréchet distance treats the intermediate curves or shapes themselves as obstacles during deformations.

Definition 4 *Given $A, B \subset M$ with $X \cong A \cong B$, the* (ambiently) isotopic Fréchet distance *between them is*

$$\mathcal{I}(A,B) = \inf_{\substack{h : M \times I \to M \\ h(\cdot,t) \textit{ homeomorphism} \\ h(x,0)=x \ \forall x \in X \\ h(A,1)=B}} \max_{x \in X} \operatorname{len} h(x,\cdot)$$

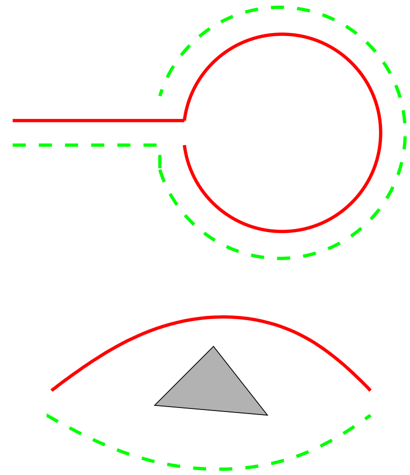


Figure 1: (a) Two curves with significantly different Hausdorff and Fréchet distances. (b) With an obstacle between them the homotopic Fréchet distance is larger than the Fréchet distance.

Ambient isotopies continuously deform both the shape and the space containing it to another shape. For example, any pair of knots in $\mathbb{R}^3$ are homotopic, but distinct knots are not ambiently isotopic, since we cannot continuously morph between them without any self-intersection along the way. This means that there are homeomorphic subsets of $\mathbb{R}^3$ that have infinite isotopic Fréchet distance.

Any ambient isotopy also defines a homotopy between A and B, so isotopic Fréchet distances is at least as large as homotopic Fréchet distance. In fact, we have

$$\mathcal{H}(A,B) \le \mathcal{F}(A,B) \le \overline{\mathcal{F}}(A,B) \le \mathcal{I}(A,B)$$

Figure 1 shows examples where Fréchet distance is strictly larger than Hausdorff distance and homotopic Fréchet distance is strictly larger than geodesic Fréchet distance. In section 5 we will give an example where homotopic and isotopic Fréchet distance differ.

Before examining any examples, we will demonstrate that it is appropriate to consider isotopic Fréchet distance a "distance".

Lemma 1 *Given any subset $X \subset M$, isotopic Fréchet distance is a metric on the space of all embeddings $X \to M$.*

Proof. Since isotopic Fréchet distance is defined as an infimum of a set of lengths of curves, it is clearly non-negative. And if the isotopic Fréchet distance is zero, then in the limit, points are moved a distance of 0. Thus $\mathcal{I}(A,B) = 0$ implies that A and B are equal.

If $h : M \times I \to M$ is a isotopy from A to B then define $h' : M \times I \to M$ by

$$h'(x,t) = h(g^{-1}(x), 1-t)$$

where $g(x) = h(x, 1)$. Clearly for any t, $h(\cdot, t)$ is a homeomorphism and

$$h'(x, 0) = h(g^{-1}(x), 1) = g(g^{-1}(x)) = x$$
$$h'(B, 1) = h(g^{-1}(B), 0) = h(A, 0) = A$$

This shows that h' is an ambient isotopy from B to A. The lengths of these two isotopies are identical so the infimum over all possible isotopies for A to B and B to A, respectively must be the same. This shows that isotopic Fréchet distance is symmetric.

Finally, we need to show that Fréchet distance satisfies the triangle inequality. Assume that $X \cong A \cong B \cong C$. It is enough to show that for any $\epsilon > 0$ there exists an isotopy $h : M \times I \to M$ from A to C such that

$$\max_{x \in X} \operatorname{len} h(x, \cdot) \leq \mathcal{I}(A, B) + \mathcal{I}(B, C) + \epsilon$$

Chose any $\epsilon > 0$. By definition of isotopic Fréchet distance there exists an isotopy $h_1 : M \times I \to M$ from A to B such that

$$\max_{x \in X} \operatorname{len} h_1(x, \cdot) \leq \mathcal{I}(A, B) + \epsilon/2$$

and an isotopy $h_2 : M \times I \to M$ from A to B such that

$$\max_{x \in X} \operatorname{len} h_2(x, \cdot) \leq \mathcal{I}(B, C) + \epsilon/2$$

Define the isotopy $h : M \times I \to M$ by

$$h(x, t) = \begin{cases} h_1(x, 2t) & \text{if } t \leq \frac{1}{2} \\ h_2(h_1(x, 1), 2t - 1) & \text{if } t > \frac{1}{2} \end{cases}$$

h is continuous since both functions agree when $t = \frac{1}{2}$. Furthermore, we see that

$$\begin{aligned} h(x, 0) &= h_1(x, 0) = x \\ h(x, 1) &= h_2(h_1(A, 1), 1) \\ &= h_2(B, 1) = C \end{aligned}$$

Thus for any $\epsilon > 0$ we have

$$\begin{aligned} \mathcal{I}(A, C) &\leq \max_{x \in X} \operatorname{len} h(x, \cdot) \\ &\leq \max_{x \in X} \operatorname{len} h_1(x, \cdot) + \max_{x \in X} \operatorname{len} h_2(x, \cdot) \\ &\leq \mathcal{I}(A, B) + \epsilon/2 + \mathcal{I}(B, C) + \epsilon/2 \\ &= \mathcal{I}(A, B) + \mathcal{I}(B, C) + \epsilon \end{aligned}$$

which completes the proof. $\square$

3 An Extra Constraint on the Isotopy

All of the distance measures between shapes which we have discussed are determined only by a single maximal distance, and many homotopies realize this distance. We can expand upon these definitions to consider the lengths of all leashes, with our end goal being to somehow minimize the distance any point travels in the homotopy realizing the minimum isotopic Fréchet distance. For any point x, the curves $h(x, \cdot)$ will be referred to as the *trajectories* of the point under the homotopy; this is also sometimes referred to as the set of *leashes*.

For a homotopy $h : X \times I \to M$ (possibly induced by an isotopy), its length function $L : X \to \mathbb{R}^+$ is defined by $L(x) = \operatorname{len} h(x, \cdot)$. Homotopic Fréchet and isotopic Fréchet distances focus on minimizing the maximum of $L(x)$ over the space of homotopies and isotopies, respectively. There are other measures of complexity, however, For example we could minimize the area or L_2 norm of these homotopies, which is equal to $\sqrt{\int_X (L(x))^2 \, dx}$, similar to what is done in [10], or we could consider some other L_p norm $\left(\int_X (L(x))^p \, dx\right)^{1/p}$. However, homotopies and isotopies minimizing these norms will not realize homotopic and isotopic Fréchet distance, respectively.

If there were only finitely many lengths to consider then we could sort them in decreasing order and then compare them. The lexicographic minimum would not only minimize the maximum of $L(x)$, but also minimize the length of the second longest leash length among homotopies minimizing the maximum. Similarly statements hold for the third longest curve and so on. However, this comparison process would only work for discrete sets. This notion can be generalized to the continuous case by consider the set of trajectories lengths for points that are local maximi of the function $L(x)$. When these sets of lengths are minimized lexicographically not only is the length of the longest curve minimized but the next largest local maximum in lengths is also minimized and so on. This yields a complexity measure on homotopies that not only realizes homotopic or Fréchet distances (depending on the space the infimum is taken over) but also moves other points as little as possible. In fact, algorithms used to compute Fréchet, geodesic Fréchet and homotopic Fréchet distances all produce pairs that minimize these more general complexities.

In 3 dimensions, minimal isotopies can be used to morph between homeomorphic shapes. Isotopies that minimize complexity would, in some sense, be minimal morphs between the two shapes. If an efficient algorithm could be found to minimize this complexity then it would yield morphs with some quality guarantees.

4 An Example

Consider the spiral curve in figure 2 compared to a straight line segment. In the minimal homotopy between them, most of the spiral collapses to a single point. This is not an ambient isotopy because at time 1 in the homotopy there is an instantaneous change in

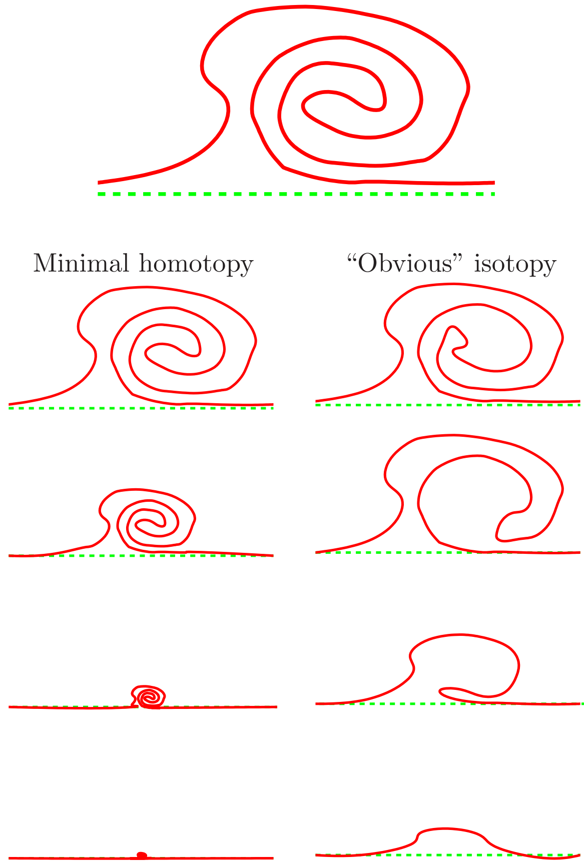


Figure 2: Comparing a spiral to a straight line. The minimal homotopy of a spiral to a line collapse the spiral to a point and the “obvious” isotopy of the spiral unravels it (note: this is not minimal).

the topology of small neighborhoods of this collapsing point.

This means that this minimal homotopy does not come from an isotopy. A natural possibility for an isotopy between the two curves is also shown in figure 2. This isotopy unravels the spiral until it flattens out completely. It is conceivable that this “obvious” isotopy realizes isotopic Fréchet distance.

In fact, for this spiral curve the isotopic Fréchet distance is equal to the Fréchet distance. To see this notice that the homotopy that realizes Fréchet distance is an isotopy arbitrarily close to time $t = 1$. So this homotopy can be followed until the spiral is as small as desired and then unwrapped. This will result in an isotopy whose longest trajectory is arbitrarily close to the Fréchet distance. This gives a sequence of isotopies whose longest trajectory length limits to the Fréchet distance proving that the two distance measures are (somewhat surprisingly) the same in this particular instance.

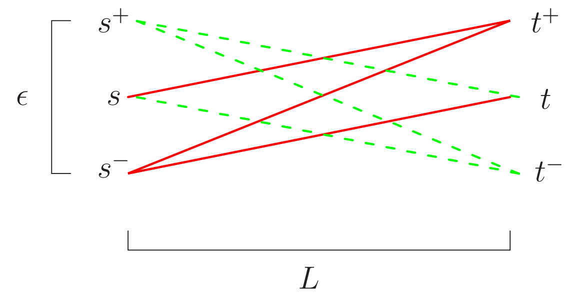


Figure 3: Two curves with Fréchet distance ϵ, but isotopic Fréchet distance at least $\frac{2}{9}L$. (Conjecturally the isotopic Fréchet distance is $\sqrt{L^2+\epsilon^2}$.)

5 Isotopic Fréchet $\neq$ Homotopic Fréchet

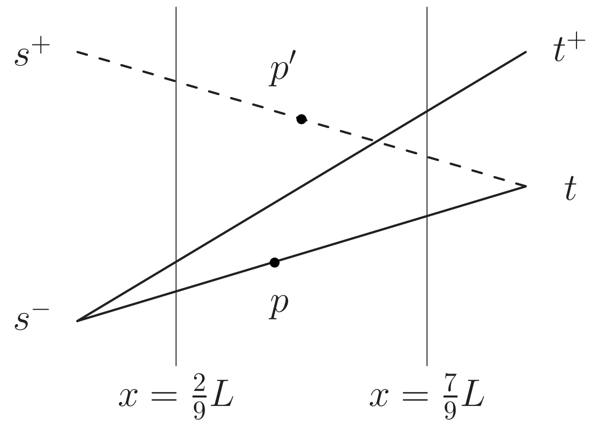
The pair of oppositely oriented “zig-zag” curves in figure 3 give an example where the curves are very close in terms of Fréchet distance but very far apart in isotopic Fréchet distance. The minimal homotopy between these curves is shown in figure 4. The homotopy preserves x coordinates of all of the points. It narrows the zig-zag until it flattens out and then expands it in the opposite direction. This is not an isotopy, and unlike the previous example, it cannot be modified to yield an isotopy. In fact, we will show that we can achieve arbitrarily large isotopic Fréchet distance relative to Fréchet distance by modifying the width and height of this figure.

Proposition 2 *For any $L > 0$ and $\epsilon \in (0, L/2)$, there exists a pair of curves $C_1, C_2 \subset \mathbb{R}^2$ with*

$$\begin{aligned} \mathcal{F}(C_1, C_2) = \mathcal{H}(C_1, C_2) &= \epsilon \\ \mathcal{I}(C_1, C_2) &\geq \frac{2}{9}L \end{aligned}$$

Proof. Consider the two curves in Figure 3, where the vertices are the points $s = (0, 0)$, $s^+ = (0, \epsilon/2)$, $s^- = (0, -\epsilon/2), t = (L, 0)$, $t^+ = (L, \epsilon/2)$ and $t^- = (L, -\epsilon/2)$. The first curve, C_1, consists of line segments $s \to t^+ \to s^- \to t$ and the second, C_2, travels from $s \to t^- \to s^+ \to t$. An easy exercise in calculating Fréchet distance shows that $\mathcal{F}(C_1, C_2) = \epsilon$. Furthermore, the maximum distances are realized by identifying t^+ to t^- and s^+ to s^-. (Note that since there are no obstacles, the Fréchet distance between these curves is the same as the homotopic Fréchet distance.)

Assume $h : M \times I \to M$ is a minimal isotopy between C_1 and C_2 and that it moves each point in C_1 along a curve whose length is at most $\frac{2}{9}L$. So we may assume that the points t^+ and s^- are moved a distance at most $\frac{2}{9}L$ by the isotopy. Let p be the point $(\frac{4}{9}L, -\frac{2}{9}\epsilon)$ on the curve C_1. Assume that after the isotopy p is sent to $p' = h(p, 1)$. If p' is not on the line segment from s^+ to t to the left of the line $x = \frac{2}{3}L$ then some point on C_1 between p and t is moved a distance greater than $\frac{2}{9}L$, a contradiction. So, we will assume that p' is on the line segment from s^+ to t with x coordinate at most $\frac{2}{3}L$.



Let l be the line segment from s^- to t^+. The point p is below l and the isotopy takes p to p' which is above l. Both the line and point move during the isotopy, but they cannot cross. The furthest we are allowing s^- to move under the isotopy is $\frac{2}{9}L$, so the x coordinate never exceeds $\frac{2}{9}L$. Similarly, the x coordinate of t^+ never drops under $\frac{7}{9}L$ as the point moves. Hence, during it's path in the isotopy p must have it's x-coordinate either go below $\frac{2}{9}L$ or above $\frac{7}{9}L$. So the length that p moves is at least L. This implies that any isotopy must move some point a distance of at least $\frac{2}{9}L$, providing the lower bound on isotopic Fréchet distance. $\square$

In figure 4, a few intermediate curves of an isotopy between the two curves are shown. This isotopy leaves s and t fixed, s^+ is sent to t^- at unit speed and s^- is sent to t^+ at unit speed. The line segments are sent to straight lines connecting these points as they move. The corners are the points that move furthest, and they move a distance of $\sqrt{L^2 + \epsilon^2}$. We conjecture that this is the isotopic Fréchet distance between the two curves. Note that in this isotopy, the trajectories of every point follows a straight line, but this will not be not true in general.

6 Calculating Isotopic Fréchet Distance

For curves in the plane, Fréchet distance can be calculated in quadratic time [1], and when polygonal obstacles are present, homotopic Fréchet distance can also be calculated in polynomial time [4]. These algorithms rely on the fact that trajectories on any point or leash must be a straight line if no obstacles are present and a geodesic in general. However, for isotopies trajectories, we must avoid other intermediate points, and so the trajectories will typically be piecewise linear. Also, isotopies do not need to proceed monotonically; in fact, they may have to back-track multiple times in the course of a minimal isotopy.

For piecewise-linear curves or surfaces, isotopic Fréchet distance can be approximated by turning it into a high dimensional motion planning problem. This approach would work for both 2 and 3 dimensional shapes. While it might yield good approximations to isotopic Fréchet distance, we would not expect these algorithms to be particularly fast.

It is also possible that previous approaches to morphing, such as [9], may yield computations that would realize the isotopic Fréchet distance, although the connection is not clear.

7 Applications

As shown above, the isotropic Fréchet distance more faithfully captures the effort of deforming one shape into another when compared to the homotopic Fréchet distance, particularly between undulating shapes. Hence it can serve as a better similarity measure between such shapes, which occur in many relevant settings such as human cortical surfaces which contain numerous folds (sulci and gyri). It could also yield a similarity measure between different structures for the same protein. Moreover, the algorithm for finding the isotropic Fréchet distance would also yield an optimal morphing sequence where each intermediate shape is free of intersections. Such intersection-free morphing is highly desirable for computer graphics applications such as animation, yet computational methods are scarce [7, 8, 9].

8 Future Work

Obviously, the most interesting open problem remaining is to determine an algorithm to compute isotopic Fréchet distance. The main challenge here is that we cannot fix obstacles in this measure, as is done in both geodesic and homotopic Fréchet distance, since the obstacles are the curves themselves as they change over time, so the problem seems harder than computing Fréchet distance between curves.

In more general spaces, not much is known beyond the fact that computing Fréchet distance between surfaces is upper semi-computable [3] and hard for some cases [2], so it is perhaps more reasonable to look for approximation algorithms to compute isotopic Fréchet distance in settings such as this.

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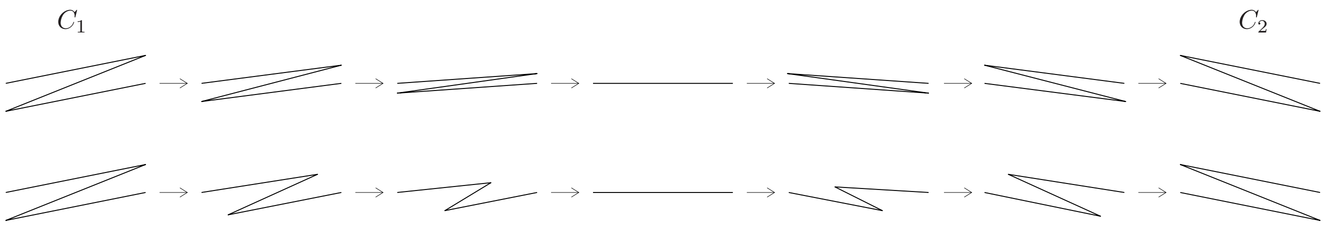

Figure 4: (a) A minimal homotopy between the curves C_1 and C_2. Note that this homotopy is not an isotopy. (b) A conjectured minimal isotopy between the two curves C_1 and C_2, this isotopy gives an upper bound on isotopic Fréchet distance of $\sqrt{L^2 + \epsilon^2}$.

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