On Inducing n-gons

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Abstract

In this paper, we establish a lower bound on the number of inducing simple n-gons in grid-like arrangements of lines. We also show that the complexity associated with counting the number of inducing n-gons in an arrangement of collinear segments is #P-complete.

1 Introduction

Arrangement of lines in the plane is among the most studied structures in combinatorial and computational geometry. Consider an arrangement of n lines. An *inducing n-gon* is a simple polygon with n sides such that extension of each side induces a line of the arrangement, and extensions of all sides induce the whole arrangement. It means that each line in the arrangement should exactly contain one side of the inducing n-gon.

An interesting question is to find out whether an arrangement includes a simple n-gon inducing the whole arrangement [3], and a more appealing question is to find an upper or lower bound on the number of these n-gons that an arrangement can tolerate [2]. The first question has been responded affirmatively for simple arrangements [1, 5] while the second one still remains open. In addition to the above problems, the complexity of counting inducing n-gons in arrangements is another appealing issue to those interested in complexity theory.

In this paper, we establish a lower bound on the number of inducing n-gons in a *grid-like arrangement of lines* as defined formally in Section 3. This class of arrangements is interesting because despite the well-shaped appearance of the arrangements it seems to be hard to count all the inducing n-gons.

Inducing n-gons are discussed in arrangements of lines and pseudo-lines [3]. In this work, the complexity of counting these n-gons in *arrangements of collinear segments* is studied.

This paper is organized as follows. In Section 2, we present a method to count a subset of inducing n-gons in a special class of arrangements. The results of Section 3 are utilized to present a lower bound on the number of inducing n-gons in grid-like arrangements of lines. In Section 4, it is shown that the complexity of counting the number of inducing n-gons in arrangements of collinear segments is #P-complete.

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2 Arrangements of n lines with at least factorial number of inducing n-gons

This section is concentrated on a specific class of arrangements of n lines, where $n = 3m$ and m is an integer. We present a method to show that an arrangement in this class contains at least factorial number of inducing n-gons.

2.1 Initialization

Consider an arrangement of $3m$ lines arranged in three sets of m parallel lines. Call the sets R, L and B, and consider the set B to be horizontal. The lines of each set are parallel to one side of an empty hypothetical triangle with horizontal base. Label the lines of each set in an ascending order, from the inner to the outer line. The intersection point between two lines is denoted by the names of those lines, x_{b_i,r_j} for the intersection of the lines b_i and r_j.

Intersections of any triple of lines, each one from different set, forms a triangle. We shall regard the area inside the biggest triangle as *arrangement-core*, and denote the triangles formed by the intersection of b_1, r_i and l_i as T_i. Call all the segments on l_is and r_is of T_is *mountain range-LR*. Similarly, all the segments on b_is and l_is of T_is and likewise all the segments on b_is and r_is of T_is are designated. We only explain the method for the mountain range-LR because of the symmetrical appearance of the arrangement.

In each mountain range, there are m *mountain*s and $m-1$ narrow *corridor*s, which play a fundamental role in the following section. Let M_i denote the ith mountain, and label the mountains and corridors from the inner to the outer one in an ascending order. Half of each corridor is on the right and the other half is on the left side of the arrangement; see Fig. 1(a). There is a specific edge in all the n-gons constructed by the method, call it *ceiling-edge*. Depending upon the parity of m, the ceiling-edge is defined differently that is described in the following section.

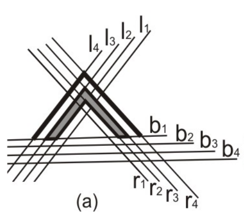

Figure 1: (a) The second corridor is filled with gray and M_4 is bolded. (b) An inducing 12-gon.

2.2 The method

All the inducing n-gons constructed by the method, contain one of the three mountain ranges. Through contribution of the segments on the mountain range-LR, there are $2m$ lines induced. Therefore, we extend the selected mountain range and close each corridor with a segment on a line of B to induce all the lines and obtain an inducing n-gon.

Assume m is even. The ceiling-edge for even m is a segment of a line of B, e.g. b_z, with two endpoints x_{b_z,r_1} and x_{b_z,r_m} or two endpoints x_{b_z,l_1} and x_{b_z,l_m}. Because of the symmetry, let us consider the ceiling-edge with two endpoints x_{b_z,r_1} and x_{b_z,r_m} on the right side, on b_z ($z \geq \lceil \frac{m}{2} \rceil$). Lemma 1 explains why it is necessary that the ceiling-edge lies on a line of B with an index greater than $\lceil \frac{m}{2} \rceil$.

Having fixed the ceiling-edge, extend the right segments of M_1 and M_m to reach it. As the ceiling-edge is placed on the right, extend $\frac{m}{2} - 1$ of the right half-corridors, and close each one of them with a segment on a so far unused line of B above b_z. Furthermore on the other side, extend $\frac{m}{2}$ of the left half-corridors and close each one with a segment on a remaining unused line of B. As presented in Fig. 1(b), the ceiling-edge is on b_3 and the extended corridors are closed on b_2, b_1, b_4 respectively. In summary, the even half-corridors are closed on the right, and the odd half-corridors are closed on the left alternatively starting from the first left half-corridor.

In conclusion, select $m-z$ odd corridors from the left to be extended bellow b_z then there are $(m-z)!$ possible choices for them to be closed on the left. Similarly, for the remaining $z-1$ half-corridors on both left and right, there are $(z-1)!$ different configurations to be extended up to above b_z and to be closed. This method uses each line exactly once and all the inducing n-gons are different.

There are $\binom{\frac{m}{2}}{m-z}(m-z)!(z-1)!$ number of inducing n-gons by fixing the ceiling-edge on b_z ($z \geq \lceil \frac{m}{2} \rceil$). Therefore, our method constructs $\sum_{\frac{m}{2}}^{m} \binom{\frac{m}{2}}{m-z}(m-z)!(z-1)!$ number of different inducing n-gons by taking all the possible places for the ceiling-edge into account, $z \geq \lceil \frac{m}{2} \rceil$. Disregarding the algebraic simplification, the result equals $(\frac{m}{2})!(\frac{m}{2}-1)![\binom{m}{\frac{m}{2}}-1]$.

The discussed points for even m are also true for odd m with some changes:

- The ceiling-edge is a segment on b_z, $z \geq \lceil \frac{m}{2} \rceil$, with two endpoints x_{b_z,l_1} and x_{b_z,r_m} or two endpoints x_{b_z,r_1} and x_{b_z,l_m}. Because of the symmetry, consider the ceiling-edge with two endpoints x_{b_z,l_1} and x_{b_z,r_m} on the right side of the arrangement. Then it is necessary to extend the left segment of M_1 and the right segment of M_m to reach the ceiling-edge, and the right half-corridors should be extended and closed above the ceiling-edge to avoid crossing it.

- Since m is odd, extend and close $\frac{m-1}{2}$ of the even corridors on the left, and do the same for $\frac{m-1}{2}$ of the odd corridors on the right alternatively starting from the first right corridor. According to the position of the ceiling-edge, whether it is on the left or on the right, some of the corridors should be closed above the line containing the ceiling-edge, and the remaining ones ought to be closed below it.

The lower bound presented for even m is also true here.

Lemma 1 *The ceiling-edge has to lie on b_z, $z \geq \lceil \frac{m}{2} \rceil$.*

Proof. By contradiction, while closing some corridors on the left or right, there would be an intersection among the ceiling-edge and the extended corridors. Therefore, there are more than one side of an inducing polygon on the ceiling-edge or other lines, and of course the result is not an inducing n-gon. It is also possible to have some self-intersections. □

2.3 Generalization of the arrangements

The presented lower bound in the previous section is preserved for generalized arrangements of n lines, where $n = km$ and k and m are both integers. Lines in the arrangements are divided into k sets of parallel lines, each set of size m. In addition, each pair of sets are intersecting, and the lines of each set are parallel to one side of an empty hypothetical k-gon. The arrangement-core for a generalized arrangement is the limited space with the exterior line of each set. To take advantage of the benefits attributed to the method discussed in Section 2.2, each set of lines should only intersect its adjacent sets inside the arrangement-core, and the intersections with other sets lie outside of this area.

It is important to note that the method in the previous section presents a lower bound on the number of inducing n-gons inside the arrangement-core, which indicates that the method ignores the intersections beyond this region. This is an observation which is used to establish a lower bound on the number of inducing n-gons in the following section.

3 Inducing n-gons in grid-like arrangements

A grid-like arrangement is an arrangement with two sets of parallel lines. If the numbers of lines in two sets are not equal, then there is no inducing n-gon. Otherwise the numbers of horizontal and vertical sides of inducing n-gons are not equal, and this cannot happen. Thus, consider a grid of size n with two sets of parallel lines each one of size m, where m is an integer. Without loss of generality, consider the lines of one set parallel to the x-axis and those of the other set parallel to the y-axis.

An obvious lower bound on the number of inducing n-gons in the grid is $(m-1)!$. This amount is obtained by considering the biggest bounded segment on the bottommost line as a fixed edge, closing each *vertical corridor* with a segment on a horizontal line and finally joining them with vertical segments on the vertical lines. The inducing n-gons obtained by this method, form monotone orthogonal n-gons in the direction of the x-axis. The former results in Section 2.2 are utilized to improve this bound.

Consider an arrangement of $2kz$ lines, the same as the arrangements described in Section 2.3, where k is a divisor of m and z is an integer. The arrangement contains $2k$ sets of parallel lines each one of size z. The $2k$ sets are arranged such that the arrangement-core forms a $2k$-gon, in which each set intersects only its adjacent sets. So it can be concluded that the segments of non-adjacent sets, inside the arrangement-core, are parallel because they do not intersect each other. It is a transformation of the grid to an arrangement of segments bounded with the arrangement-core. In other words, the whole m lines in one set of the grid are divided into k sets in such a way that each set contains $\frac{m}{k}$ lines. These k sets are arranged parallel to the non-consecutive sides of a hypothetical $2k$-gon.

Although a grid contains extra intersections in comparison with the arrangement of segments inside the arrangement-core, simply ignore those additional intersections; see Fig. 2(a). In other words, it is a lower bound on the number of inducing n-gons where the n-gons do not bend on the extra intersections; see Fig. 2(b).

Based on the above discussion, the presented lower bound in Section 2.2 is also true for grids. As there is no overlap between inducing n-gons obtained by the two methods, the lower bound is equal to $(m-1)!+\sum_{k\in K}(\frac{m}{2k})!(\frac{m}{2k}-1)![\binom{\frac{m}{k}}{\frac{m}{2k}}-1]$, where K is the set of all divisors of m which are greater than two. Note that for a fixed k, the ceiling-edge contains exactly $k-1$ segments. Therefore, different k do not lead to identical n-gons while there is at least one difference between their ceiling-edges. This bound can become more precise by taking inducing n-gons in other directions into account although we ignore them.

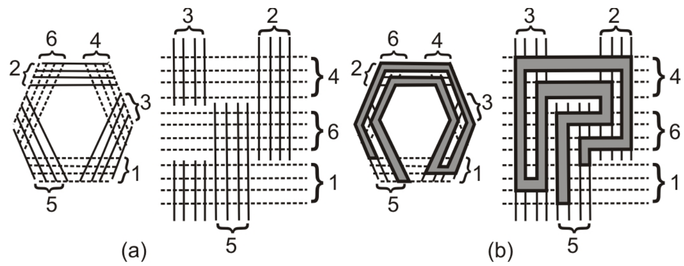

Figure 2: (a) Arrangement of segments inside the arrangement-core and the related grid, extra intersections are removed. (b) The related inducing n-gons.

4 Complexity of counting inducing n-gons in an arrangement of collinear segments

An **a**rrangement of **c**ollinear **s**egments, ACS, is a collection of line segments in the plane. The arrangement includes some maximal subsets of collinear segments, i.e. a maximal subset *family*. A single segment can also be a family if there is no other segment collinear with it. Note that an arrangement of lines is a special case for this arrangement, as there are n families in an arrangement of n lines.

An inducing n-gon in ACS is a polygon with n sides for which there is a bijective relation between its sides and the families. It means that each family of ACS should contain exactly one side of the inducing n-gon. Obviously, if there are less than n intersections between the families of ACS, there is no inducing n-gon.

The class #P contains all counting problems associated with the polynomial-balanced and polynomial-time decidable relations [4]. As our problem satisfies these two properties, it is in #P. We demonstrate that counting the number of inducing n-gons in an arrangement of collinear segments is #P-complete. Let us reduce the #P-complete problem *#RPM* to *#n-IP*, where #RPM is the number of perfect matchings in a regular bipartite graph, and #n-IP is the number of inducing n-gons in an ACS.

Theorem 2 *Complexity of counting #RPM in a k-regular bipartite graph is #P-complete, for any fixed $k>2$ [6] .*

Given a k-regular bipartite graph $G=(U,V,E)$ such that $|U|=|V|=m$. The goal is to construct an arrangement of collinear segments such that the inducing n-gons in ACS somehow correspond to the perfect matchings in G. Consider some *guide-lines* which form a $(m+1)$ times $(m+1)$ grid-like arrangement A, and also suppose m vertical *guide-segments* such that each segment is limited to the second bottommost guide-line and a point above the topmost horizontal guide-line. Each guide-segment is placed inside a vertical corridor of A and divides it into two vertical corridors, a *small* *corridor* and a *big corridor*. For each node in U and V consider a big corridor and a horizontal guide-line of A respectively. See Fig. 3(a) and consider the following segments:

- *Main-segments*: Bounded horizontal segments with big corridors.
- *Small-segments*: Bounded horizontal segments with small corridors.
- *Helping-segments*: The segments on both guide-segments and vertical guide-lines bounded between the second bottommost horizontal guide-line and the topmost small segment.
- *Final-segments*: The biggest segment on the bottommost guide-line and the connecting segments of its endpoints to the left and right most helping-segments.

The reduction is as follows. For each edge in G which connects u_i to v_j, add a main-segment inside the big corridor associated with u_i and lies on the guide-line attributed to v_j, $1 \leq i, j \leq m$. Add m small-segments inside of each small corridor. Put one of the small-segments above the topmost horizontal guide-line, and each horizontal corridor of A should contain exactly one small-segment except the bottommost corridor. The small-segments in each horizontal corridor of A should be collinear, to form horizontal families. Add the helping-segments and the final-segments; see Fig. 3(b) as overall arrangement. We claim that $\#n$-IP is equal to $m!$ #RPM, where $n = 4m + 2$. It can be shown by the following lemma.

Lemma 3 *All the inducing n-gons in the designed ACS are monotone in the direction of the x-axis.*

Proof. By contradiction, there are more than one side of the polygon on at least one of the helping-segments. □

In each inducing n-gon there are exactly m main-segments. There is no pair of main-segments selected in a big corridor as an inducing n-gon in the constructed ACS is monotone, Lemma 3. This point indicates that there is no pair of vertices in V matched with a vertex in U; see Fig. 3(c).

To obtain an inducing n-gon, the main-segments are joining via small-segments, helping-segments and final-segments. For a fixed set of selected main-segments in an inducing n-gon, related to a perfect matching of G, there are $m!$ possible ways to choose small-segments to obtain $m!$ different inducing n-gons.

The reduction is now complete and obviously is in P. It is to mean that if someone can obtain the $\#n$-IP efficiently, he could easily find the #RPM which is a #P-complete.

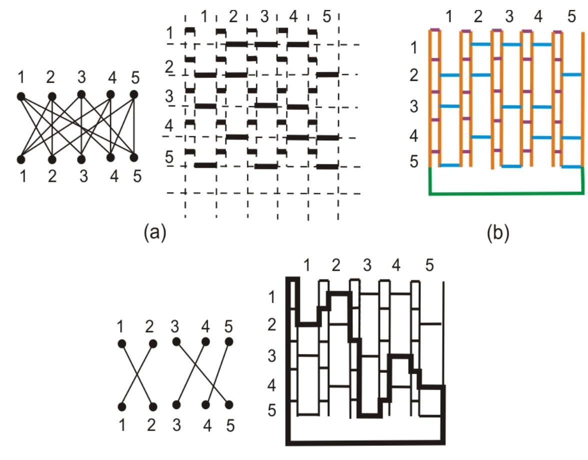

Figure 3: (a) On the left, a 3-regular bipartite graph. On the right, dash lines/segments are as guidance and the related main-segments and small-segments to the graph are bolded. (b) The main-segments, small-segments, helping-segments and final-segments are blue, violet, orange and green respectively. (c) On the left, a perfect matching. On the right, the designed arrangement of segments and a related inducing n-gon to the perfect matching is bolded.

5 Conclusion

We establish a lower bound on the number of inducing n-gons in grid-like arrangements. It is interesting to find the exact number of inducing n-gons in this class of arrangements. We also demonstrate that the complexity of counting the number of inducing n-gons in an arrangement of collinear segments is complete for the class #P. We conjecture that the complexity of counting inducing n-gons in arrangements of lines is also #P-complete.

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