# Toward the Tight Bound of the Stretch Factor of Delaunay Triangulations*

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## Abstract

In this paper, we investigate the tight bound of the stretch factor of the Delaunay triangulation by studying the stretch factor of the chain (Xia 2011). We define a sequence $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ where $\Gamma_i$ is the maximum stretch factor of a chain of $i$ circles, and show that $\Gamma$ is strictly increasing. We then present an improved lower bound of 1.5932 for the stretch factors of the Delaunay triangulation. This bound is derived from a sequence of chains sharing a set of properties. We conjecture that these properties are also shared by a chain with the worst stretch factor.


## 1 Introduction

Let $S$ be a finite set of points in the Euclidean plane. A *Delaunay triangulation* of $S$ is a triangulation in which the circumscribed circle of every triangle contains no point of $S$ in its interior. An alternative equivalent definition is: An edge $xy$ is in the Delaunay triangulation of $S$ if and only if there exists a circle through points $x$ and $y$ whose interior is devoid of points of $S$. A Delaunay triangulation of $S$ is the dual graph of the Voronoi diagram of $S$.

Let $D$ be a Delaunay triangulation of $S$. For two points $p$ and $q$ in $S$, denote by $d(p, q)$ the length of the shortest path from $p$ to $q$ following the edges of the triangles in $D$ and by $||pq||$ the Euclidean line distance between $p$ and $q$. Then the *stretch factor* (also known as *dilation* or *spanning ratio*) of a Delaunay triangulation of $S$ is the maximum value of $d(p, q)/||pq||$ over all pairs of points $p, q$ in $S$.

Proving the tight bound for the stretch factor of Delaunay triangulations has been a long standing open problem in computational geometry, with important applications in areas such as wireless communications. The stretch factor of Delaunay triangulations has an obvious lower bound of $\pi/2 \approx 1.571$ [3], which occurs when the points lie on a circle whose diameter is $pq$. Recently, Bose et. al. [2] gave an improved lower bound of $1.581 > \pi/2$ by constructing a configuration where the points are distributed on the boundary of two half circles separated by a small distance. They also showed a slightly better lower bound of 1.5846. In term of upper bounds, Dobkin, Friedman, and Supowit [5] in 1987 showed that the stretch factor of the Delaunay triangulation is at most $(1 + \sqrt{5})\pi/2 \approx 5.08$. This upper bound was improved by Keil and Gutwin [6] in 1989 to $2\pi/(3\cos(\pi/6)) \approx 2.42$. For the special case when the point set $S$ is in convex position, Cui, Kanj and Xia [4] proved that the Delaunay triangulation of $S$ has stretch factor at most 2.33. Recently, Xia [7] proposed to study the stretch factor of the Delaunay triangulation by focusing on the stretch factor of a chain of circles in the plane. With this approach, Xia [7] proved that the stretch factor of the Delaunay triangulation is less than 1.998.

Following the same approach as that in [7], we investigate the tight bound on the stretch factor of the Delaunay triangulation by studying the stretch factor of the chain. We define a sequence $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ where $\Gamma_i$ is the maximum stretch factor of a chain of $i$ circles. We prove that $\Gamma$ is strictly increasing, which implies that the tight bound of the stretch factor of the chain is the limit of $\Gamma$. We then proceed to investigate what kind of chains achieve the stretch factors in $\Gamma$. To that end, we define a family of chains $\mathbb{C} = \{\mathcal{C}_3, \mathcal{C}_5, \mathcal{C}_7, \ldots\}$, each having odd number of circles and satisfying certain structural properties. We conjecture that for all odd numbers $n \geq 3$, the stretch factor of the chain $\mathcal{C}_n \in \mathbb{C}$ is $\Gamma_n$. If this conjecture is true, then the problem of finding the tight bound is reduced to computing the limit of the stretch factor of the chains in $\mathbb{C}$.

Even without proving the conjecture, studying the stretch factor of the chains in $\mathbb{C}$ is still interesting. It yields improved lower bound. To illustrate this, we compute the chains $\mathcal{C}_n \in \mathbb{C}$ for $n = 3, 5, 7, \ldots, 31$. This yields a lower bound of 1.5932 for the stretch factors of the Delaunay triangulation, improving the previous lower bound of 1.5846 by Bose et al. [2].

The paper is organized as follows. The necessary definitions are given in Section 2. In Section 3, we show that $\Gamma$ is strictly increasing by proving that one can always increase the stretch factor of a chain by adding a circle to it. In Section 4, we present an improved lower bound of the stretch factors of the Delaunay triangulation. We conclude the paper in Section 5 with a conjecture and a question that are key to finding the tight bound of the

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stretch factor of the Delaunay triangulation.

## 2 Preliminaries

We label the points in the plane by lower case letters, such as $p, q, u, v$, etc. For any two points $p, q$ in the plane, denote by $pq$ a line in the plane passing through $p$ and $q$, by $\overline{pq}$ the line segment connecting $p$ and $q$, and by $\overrightarrow{pq}$ the ray from $p$ to $q$. The Euclidean distance between $p$ and $q$ is denoted by $||pq||$. The length of a path $P$ in the plane is denoted by $|P|$. Any angle denoted by $\angle poq$ is measured from $\overrightarrow{op}$ to $\overrightarrow{oq}$ in the *counterclockwise* direction. Unless otherwise specified, the angles are defined in the range $(-\pi, \pi]$.

**Definition 1** We say that a sequence of distinct finite circles[1] $\mathcal{C} = (O_1, O_2, \ldots, O_n)$ in the plane is a *chain*[2] if it has the following three properties. **Property (1)**: Every two consecutive circles $O_i, O_{i+1}$ intersect, $1 \leq i \leq n-1$. Let $a_i$ and $b_i$ be the shared points on their boundary (in the special case where $O_i, O_{i+1}$ are tangent, $a_i = b_i$). Without loss of generality, assume $a_i$'s are on one side of $\mathcal{C}$ and $b_i$'s are on the other side. Denote by $C_i^{(i+1)}$ the arc on the boundary of $O_i$ that is in $O_{i+1}$, and by $C_{i+1}^{(i)}$ the arc on the boundary of $O_{i+1}$ that is in $O_i$. We refer to $C_i^{(i+1)}$ and $C_{i+1}^{(i)}$ as "connecting arcs". **Property (2)**: The connecting arcs $C_i^{(i-1)}$ and $C_i^{(i+1)}$ on the same circle $O_i$ do not overlap, for $2 \leq i \leq n-1$; i.e., $C_i^{(i-1)}$ and $C_i^{(i+1)}$ do not share any point other than a boundary point. **Property (3)**: There is a ray $\overrightarrow{r}$ that crosses line segments $\overline{a_i b_i}$ for all $1 \leq i \leq n-1$ in that order. Let $u$ be the entry-point of $\overrightarrow{r}$ on $O_1$ and $v$ the exit-point of $\overrightarrow{r}$ on $O_n$. We call $u, v$ *a pair of terminal points* (or simply *terminals*) of the chain $\mathcal{C}$. See Figure 1 for an illustration.

For notational convenience, define $a_0 = b_0 = u$ and $a_n = b_n = v$. Every circle $O_i$ has two arcs on its boundary between the line segments $\overline{a_{i-1}b_{i-1}}$ and $\overline{a_i b_i}$, denoted by $A_i$ and $B_i$. Without loss of generality, assume that $a_{i-1}, a_i$ are the ends of $A_i$ and $b_{i-1}, b_i$ are the ends of $B_i$, for $1 \leq i \leq n$. This means that $P_A = A_1 \ldots A_n$ is a path from $u$ to $v$ on one side of the chain and $P_B = B_1 \ldots B_n$ is a path from $u$ to $v$ on the other side of the chain. An arc $A_i$ or $B_i$ may degenerate to a point, in which case $a_{i-1} = a_i$ or $b_{i-1} = b_i$, respectively.

We define the shortest path between $u$ and $v$ in $\mathcal{C}$, denoted by $P_{\mathcal{C}}(u, v)$, to be the shortest path from $u$ to $v$ while traveling along arcs in $\{A_1, \ldots, A_n\} \cup \{B_1, \ldots, B_n\}$ and line segments in $\{\overline{a_1 b_1}, \ldots, \overline{a_{n-1} b_{n-1}}\}$. Its length, $|P_{\mathcal{C}}(u, v)|$, is the total length of the edges in $P_{\mathcal{C}}(u, v)$. For example, in Figure 1, $P_{\mathcal{C}}(u, v)$ is the shortest path from $u$ to $v$ while traveling along the thick arcs and lines.

[1] In this paper, a circle is considered to be a closed disk in the plane

[2] Note that our definition of a chain is slightly different from the chain defined in [7]. Our chain has an additional Property (3).

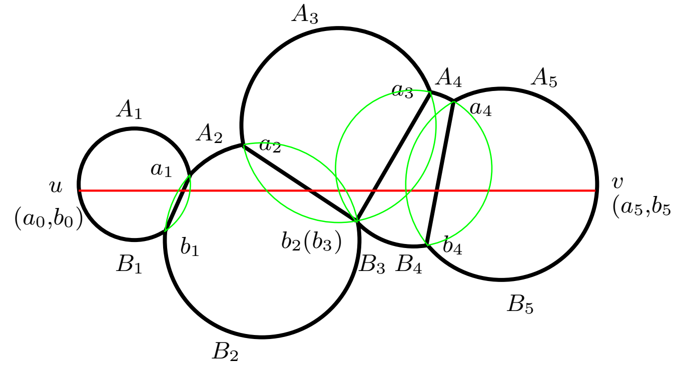



Figure 1: A chain $\mathcal{C}$. The connecting arcs are green (gray in black and white printing). The connecting arcs on the boundary of the same circle are disjoint. Points $u$ and $v$ are a pair of terminals of $\mathcal{C}$.

Now we can define the *stretch factor* of a chain $\mathcal{C}$ to be the maximum value of

$$|P_{\mathcal{C}}(u, v)| / ||uv||, \tag{1}$$

over all terminals $u, v$ of $\mathcal{C}$. The stretch factor of a chain is analogous to that of a Delaunay triangulation. From [7], the maximum stretch factor of the chain is an upper bound of the maximum stretch factor of the Delaunay triangulation. We believe that these two quantities are in fact equal (we discuss this in details in Section 4).

## 3 On the Tight Bound

Let $\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3, \ldots)$ be a sequence where $\Gamma_i$ is the maximum stretch factor of a chain of $i$ circles. In this section we show that $\Gamma$ is strictly increasing. This implies that the tight bound of the stretch factor of the chain is the limit of $\Gamma$: $\lim_{i \to \infty} \Gamma_i$.

**Theorem 2** *For all* $n \geq 1$, $\Gamma_{n+1} > \Gamma_n$.

**Proof.** Let $\mathcal{C}$ be a chain with stretch factor $\rho = \Gamma_n$. Without loss of generality, assume that $\mathcal{C}$ has the *minimum* number of circles among all chains whose stretch factor is $\Gamma_n$. We will add a circle $O_{n+1}$ to $\mathcal{C}$ such that the new chain $\mathcal{C}'$ has a stretch factor $> \rho$.

Refer to Figure 2. Let $u, v$ be terminals of $\mathcal{C}$ with stretch factor $\rho$. Without loss of generality, assume that $a_{n-1}$ is above $uv$ and $b_{n-1}$ is below $uv$. By flipping around $uv$, we can assume that $o_n$ is on or below $uv$. We can also assume that $v$ is not on the boundary of $O_{n-1}$ because otherwise, we can remove $O_n$ from $\mathcal{C}$ and still have the same stretch factor—a contradiction to

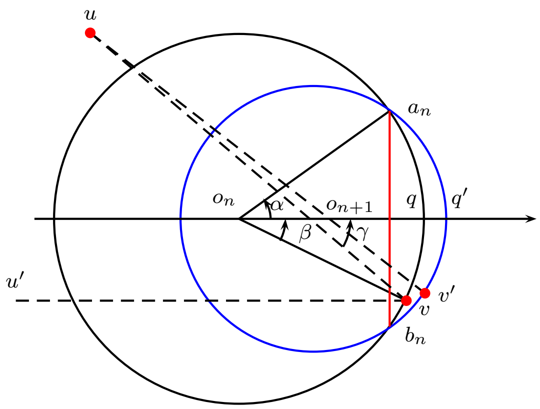



Figure 2: Illustration of adding a new circle $O_{n+1}$ to $\mathcal{C}$.

the fact that $\mathcal{C}$ is a chain of minimum number of circles with stretch factor $\rho$.

This means that there exist two points $a_n$ and $b_n$ sufficiently close to $v$ on the boundary of $O_n$ such that

$$|\widehat{va_n}| - |\widehat{vb_n}| = ||a_n b_n||, \tag{2}$$

where $\widehat{va_n}$ (resp. $\widehat{vb_n}$) is the arc on the boundary of $O_n$ between $v$ and $a_n$ (resp. between $v$ and $b_n$).

Add a new circle $O_{n+1}$ going through $a_n$ and $b_n$ whose center is $o_{n+1}$. Denote the new chain by $\mathcal{C}'$. Let $v'$ be the point on $O_{n+1}$ such that

$$|\widehat{v'a_n}| - |\widehat{v'b_n}| = ||a_n b_n||, \tag{3}$$

where $\widehat{v'a_n}$ (resp. $\widehat{v'b_n}$) is the arc on the boundary of $O_{n+1}$ between $v'$ and $a_n$ (resp. between $v'$ and $b_n$).

Refer to Figure 2. Let $\alpha$ be the angle from $\overrightarrow{o_n o_{n+1}}$ to $\overrightarrow{o_n a_n}$, $\beta$ the angle from $\overrightarrow{o_n v}$ to $\overrightarrow{o_n o_{n+1}}$, and $\gamma$ the angle from $\overrightarrow{uv}$ to $\overrightarrow{o_n o_{n+1}}$.

Let $\Delta_O = ||o_n o_{n+1}||$, $\Delta_P = |P_{\mathcal{C}'}(u, v')| - |P_{\mathcal{C}}(u, v)|$, and $\Delta_D = ||uv'|| - ||uv||$. By a standard, if lengthy, calculation, we have the following from [7]:

$$\lim_{\Delta_O \to 0} \frac{\Delta_P}{\Delta_O} = \sin\alpha - \alpha\cos\alpha, \tag{4}$$

and

$$\lim_{\Delta_O \to 0} \frac{\Delta_D}{\Delta_O} = \cos\gamma - \cos\alpha(\cos(\beta-\gamma) + \beta\sin(\beta-\gamma)). \tag{5}$$

Let $r_n$ be the radius of $O_n$. Note that $|\widehat{va_n}| = (\alpha + \beta)r_n$, $|\widehat{vb_n}| = (\alpha - \beta)r_n$, and $||a_n b_n|| = 2\sin\alpha r_n$. From (2), we have $(\alpha+\beta)r_n - (\alpha-\beta)r_n = 2\sin\alpha r_n$. This means $\beta = \sin\alpha$. We have

$$\begin{aligned}\lim_{\Delta_O \to 0} \frac{\Delta_P}{\Delta_D} &= \frac{\sin\alpha - \alpha\cos\alpha}{\cos\gamma - \cos\alpha(\cos(\beta-\gamma) + \beta\sin(\beta-\gamma))} \\ &= \frac{\sin\alpha - \alpha\cos\alpha}{\cos\gamma - \cos\alpha(\cos(\sin\alpha-\gamma) + \sin\alpha\sin(\sin\alpha-\gamma))}.\end{aligned} \tag{6}$$

Set $\alpha$ small enough, say $\alpha = 0.01$. Then

$$\lim_{\Delta_O \to 0} \frac{\Delta_P}{\Delta_D} \bigg| \{\alpha = 0.01, \gamma = 0\} > 66. \tag{7}$$

Refer to Figure 2. Let $\Delta_V = ||vv'||$. When $\gamma = 0$, we have

$$\Delta_D \mid \{\gamma = 0\} = -\cos(\angle v'vu')\Delta_V. \tag{8}$$

When $\gamma > 0$, we have

$$\Delta_D \mid \{\gamma > 0\} = -\cos(\angle v'vu)\Delta_V. \tag{9}$$

Let $q$ and $q'$ be the exit-point of $\overrightarrow{o_n o_{n+1}}$ on the boundary of $O_n$ and $O_{n+1}$, respectively. Then $|\widehat{vq}| = ||a_n b_n||/2 = |\widehat{v'q'}|$, where $\widehat{vq}$ is the arc between $v$ and $q$ on the boundary of $O_n$ and $\widehat{v'q'}$ is the arc between $v'$ and $q'$ on the boundary of $O_{n+1}$. Since $\alpha$ is small and $\Delta_O \to 0$, we have $r_n > r_{n+1}$. It is easy to see that $0 < \angle v'vu < \angle v'vu'$. Also $\angle v'vu' < \pi$ because the distance from $v'$ to line $o_n o_{n+1}$ is less than the distance from $v$ to $o_n o_{n+1}$. This means $-\cos(\angle v'vu) < -\cos(\angle v'vu')$. From (8) and (9), we have

$$\Delta_D \mid \{\gamma > 0\} < \Delta_D \mid \{\gamma = 0\}.$$

Therefore

$$\begin{aligned}&\lim_{\Delta_O \to 0} \frac{\Delta_P}{\Delta_D} \bigg| \{\alpha = 0.01, \gamma > 0\} \\ &> \lim_{\Delta_O \to 0} \frac{\Delta_P}{\Delta_D} \bigg| \{\alpha = 0.01, \gamma = 0\} \\ &> 66.\end{aligned} \tag{10}$$

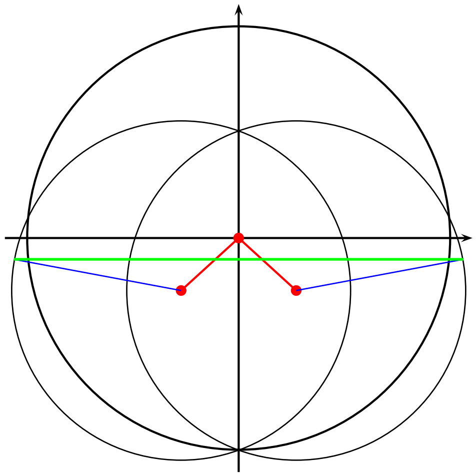
Since $o_n$ is on or below $uv$, $\gamma > 0$. When $\Delta_O \to 0$ and $\alpha$ small enough, we have $\frac{\Delta_P}{\Delta_D} > 66$. From [7], the stretch factor of the chain is less than 2. So $\frac{|P_{\mathcal{C}}(u,v)|}{||uv||} = \rho < 2$. Therefore, we have

$$\frac{|P_{\mathcal{C}'}(u, v')|}{||uv'||} = \frac{|P_{\mathcal{C}}(u,v)| + \Delta_P}{||uv|| + \Delta_D} > \frac{|P_{\mathcal{C}}(u,v)|}{||uv||} = \rho.$$

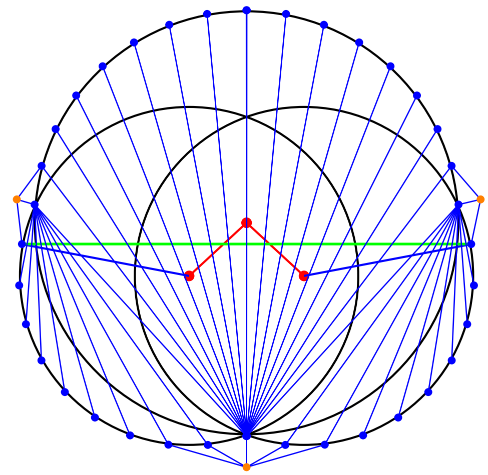
This completes the proof. $\square$

## 4 Improved Lower bounds

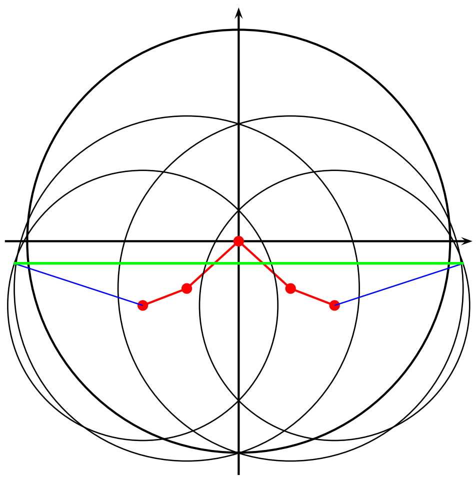
A natural question is what kind of chains achieve the worst stretch factor $\Gamma_n$. We present chains of 3, 5, 7, 15, 31 circles (see Figure 3) with stretch factors 1.5894,



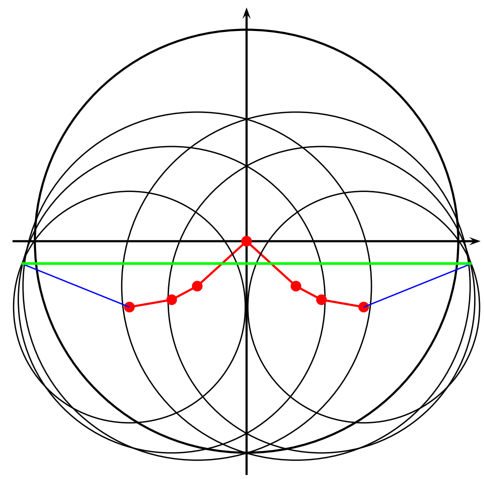
(a) A chain a 3 circles with stretch factor $\rho_3 \approx 1.5894$.



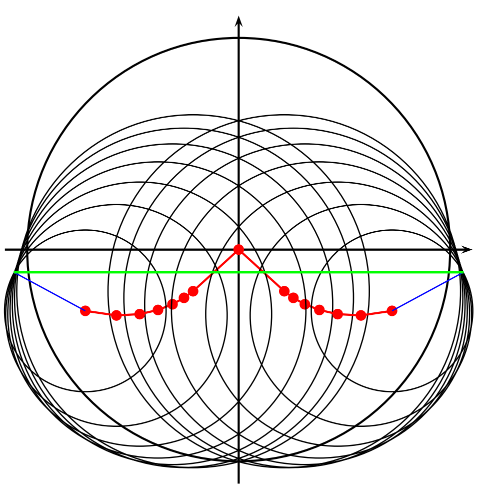
(b) A Delaunay triangulation based on a chain of 3 circles with the same stretch factor as $\rho_3 \approx 1.5894$. The orange "shield" points are added to prevent short-cuts outside of the chain.



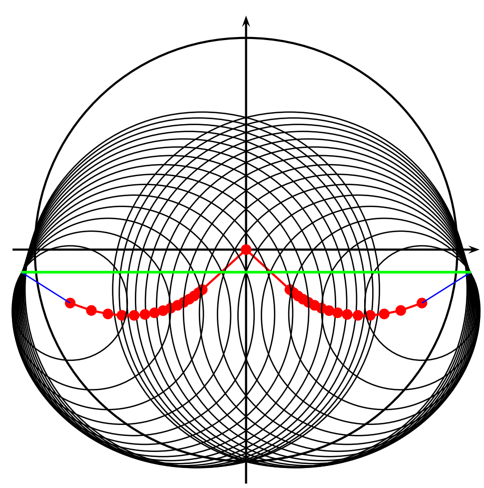
(c) A chain of 5 circles with stretch factor $\rho_5 \approx 1.5919$.



(d) A chain of 7 circles with stretch factor $\rho_7 \approx 1.5926$.



(e) A chain of 15 circles with stretch factor $\rho_{15} \approx 1.5931$.



(f) A chain of 31 circles with stretch factor $\rho_{31} \approx 1.5932$.

Figure 3: Illustration of the chains with improved lower bounds. The green line connects terminals $u$ and $v$. The red dots are the centers of the circles in the chain. The blue lines in (b) show the Delaunay triangulation.

1.5919, 1.5926, 1.5931, and 1.5932, respectively. The exact configuration of the chain of 31 circles is given in the Appendix.

For each chain given in Figure 3, we can create a Delaunay triangulation with the same stretch factor as follows: place points of $S$ densely on the outer boundary of $\mathcal{C}_n$. With a small perturbation, one can ensure that the edges of the Delaunay triangulation inside the chain do not provide a short-cut for any shortest path between the terminals $u$ and $v$ in the chain, as shown in Figure 3(b). In order to prevent short-cuts outside of the chains, we use the technique of Bose et al. [2] by adding "shield" points, shown as the orange points in Figure 3(b). This yields a lower bound of 1.5932 on the stretch factor of the Delaunay triangulation, improving the previous best lower bound of 1.5846 by Bose et al. [1].

**Theorem 3** *There exists a set $S$ of points in the plane, such that the Delaunay triangulation of $S$ has a stretch factor of at least 1.5932.*

## 5 Toward the Tight Bound

The chains $\mathcal{C}_n$ in Figure 3 all share the following properties: let $n = 2k + 1$ and let $u$ and $v$ be the terminals of $\mathcal{C}_n$, then

1. for all $1 \le i \le k$, $O_{k+1+i}$ and $O_{k+1-i}$ are symmetric around a line $l$ passing through $o_{k+1}$, the center of $O_{k+1}$,
2. $O_k$, $O_{k+1}$ and $O_{k+2}$ share a point on $l$,
3. the radii of $O_{k+1}, O_{k+2}, \ldots, O_{2k+1}$ are in decreasing order and the radii of $O_1, O_2, \ldots, O_{k+1}$ are in increasing order,
4. for any $1 \le i \le n-1$, $\overline{a_i b_i}$ is contained in a shortest path from $u$ to $v$, and
5. both $P_A = A_1 \ldots A_n$ and $P_B = B_1 \ldots B_n$ are shortest paths from $u$ to $v$.

We conjecture that these properties are shared by a chain with the worst stretch factor:

**Conjecture 4** *For all $n = 2k + 1 \ge 3$ there is a chain of $n$ circles with stretch $\Gamma_n$ that satisfies Properties 1–5.*

Note that we can assume Property 5 is true because of the following observation.

**Proposition 5 ([7])** *There is a chain $\mathcal{C}^*$ of $n$ circles with stretch factor $\Gamma_n$, in which both $P_A$ and $P_B$ are shortest paths.*

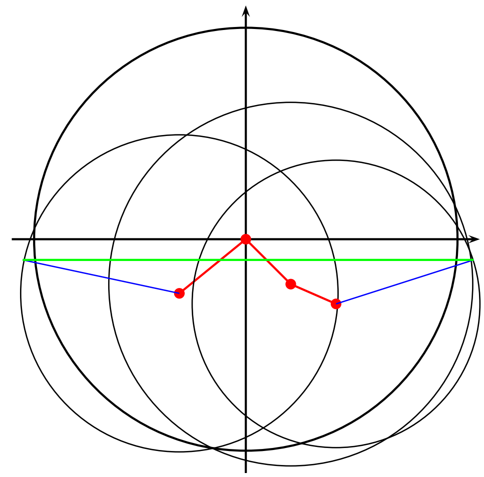


Figure 4: A chain of 4 circles with stretch factor 1.5907.

Proposition 5 was proved in [7] using a technique that transforms any chain of $n$ circles into $\mathcal{C}^*$ without reducing the stretch factor. A similar technique of transformation may be helpful in proving other properties.

If Conjecture 4 is true, the task of finding the tight bound of the stretch factor of the Delaunay triangulation is reduced to answering the following question.

> **Question:** What is the worst stretch factor of a chain satisfying Conditions 1–5.

Even without proving Conjecture 4, the answer to this question will give an improved lower bound of the stretch factor of the Delaunay triangulation.

Finally, note that a chain of even number of circles with the maximum stretch factor may not have the symmetry exhibited by the chains of odd number of circles. See Figure 4 for a chain of 4 circles with stretch factor 1.5907. This chain is not symmetric and all circles in it have different sizes.

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## Appendix

In the following, we give the exact configurations of chains of 31 circles, as shown in Figure 3 (f). Let $n = 31$. $\rho_n$ is the stretch factor, $b_1$ is the angle from $\overrightarrow{o_1u}$ to $\overrightarrow{o_2o_1}$, and $b_n$ is the angle from $\overrightarrow{o_{n-1}o_n}$ to $\overrightarrow{o_nv}$. For all $1 \le i \le n$, $(x_i, y_i)$ are the $x$- and $y$-coordinates of $o_i$—the center of $O_i$, and $r_i$ is the radius of $O_i$.

$$\begin{aligned}
\rho_{31} &= 1.59321532337905\\
(x_1, y_1) &= (-82.83285023949975, -25.121488078241036)\\
r_1 &= 27.2227454174619\\
(x_2, y_2) &= (-72.85751097488247, -28.607891622305775)\\
r_2 &= 37.5929692832941\\
(x_3, y_3) &= (-65.06105288129035, -30.244664465966718)\\
r_3 &= 45.2705987926872\\
(x_4, y_4) &= (-58.46281391202502, -30.896769916666095)\\
r_4 &= 51.5306676497054\\
(x_5, y_5) &= (-52.71657386093427, -30.908699522242472)\\
r_5 &= 56.8317425129397\\
(x_6, y_6) &= (-47.641009930136384, -30.47042857093014)\\
r_6 &= 61.4105580627107\\
(x_7, y_7) &= (-43.12496351134913, -29.70369563073232)\\
r_7 &= 65.4084222645816\\
(x_8, y_8) &= (-39.09176839012416, -28.69398435657018)\\
r_8 &= 68.9185983263288\\
(x_9, y_9) &= (-35.4841587781849, -27.505033297273425)\\
r_9 &= 72.0068096919405\\
(x_{10}, y_{10}) &= (-32.25667996954523, -26.186384074676056)\\
r_{10} &= 74.7216529992949\\
(x_{11}, y_{11}) &= (-29.371513326910353, -24.777713407816105)\\
r_{11} &= 77.1003848112902\\
(x_{12}, y_{12}) &= (-26.795869450721117, -23.311439841674048)\\
r_{12} &= 79.1724821194007\\
(x_{13}, y_{13}) &= (-24.50025982738938, -21.814336462991804)\\
r_{13} &= 80.9619450886396\\
(x_{14}, y_{14}) &= (-22.457239646974063, -20.308501263020666)\\
r_{14} &= 82.488877282518\\
(x_{15}, y_{15}) &= (-20.639566299045978, -18.8111754808014)\\
r_{15} &= 83.7712177530083\\
(x_{16}, y_{16}) &= (0.0, 0.0)\\
r_{16} &= 100.0\\
(x_{17}, y_{17}) &= (20.639566299045978, -18.8111754808014)\\
r_{17} &= 83.7712177530083\\
(x_{18}, y_{18}) &= (22.457239646974063, -20.308501263020666)\\
r_{18} &= 82.488877282518\\
(x_{19}, y_{19}) &= (24.50025982738938, -21.814336462991804)\\
r_{19} &= 80.9619450886396\\
(x_{20}, y_{20}) &= (26.795869450721117, -23.311439841674048)\\
r_{20} &= 79.1724821194007\\
(x_{21}, y_{21}) &= (29.371513326910353, -24.777713407816105)\\
r_{21} &= 77.1003848112902\\
(x_{22}, y_{22}) &= (32.25667996954523, -26.186384074676056)\\
r_{22} &= 74.7216529992949\\
(x_{23}, y_{23}) &= (35.4841587781849, -27.505033297273425)\\
r_{23} &= 72.0068096919405\\
(x_{24}, y_{24}) &= (39.09176839012416, -28.69398435657018)\\
r_{24} &= 68.9185983263288\\
(x_{25}, y_{25}) &= (43.12496351134913, -29.70369563073232)\\
r_{25} &= 65.4084222645816\\
(x_{26}, y_{26}) &= (47.641009930136384, -30.47042857093014)\\
r_{26} &= 61.4105580627107\\
(x_{27}, y_{27}) &= (52.71657386093427, -30.908699522242472)\\
r_{27} &= 56.8317425129397\\
(x_{28}, y_{28}) &= (58.46281391202502, -30.896769916666095)\\
r_{28} &= 51.5306676497054\\
(x_{29}, y_{29}) &= (65.06105288129035, -30.244664465966718)\\
r_{29} &= 45.2705987926872\\
(x_{30}, y_{30}) &= (72.85751097488247, -28.607891622305775)\\
r_{30} &= 37.5929692832941\\
(x_{31}, y_{31}) &= (82.83285023949975, -25.121488078241036)\\
r_{31} &= 27.2227454174619\\
b_1 = b_{31} &= 0.22563636621218
\end{aligned}$$