On covering of any point configuration by disjoint unit disks

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Abstract

We give a configuration of 53 points that cannot be covered by disjoint unit disks. This improves the previously known configuration of 55 points.

1 Introduction

In 2008, Japanese puzzle designer Naoki Inaba proposed an interesting question [3]: "Given any configuration of 10 points, prove that you can cover all the points by coins. You can use any number of coins, but coins cannot overlap." That is, he proved the following theorem:

Theorem 1 *Any configuration of 10 points can be covered by disjoint coins.*

Inaba gave an interesting proof of this theorem based on the probabilistic method. (See appendix; this proof is essentially the same in [4] written in Japanese. The proof can be found in [6] also.) As he mentioned in the answer page [4], this theorem also derives another natural question: How many points arranged appropriately cannot be covered by disjoint coins? Let k be the maximum number of points such that any configuration of k points can be covered by coins. (We note that k points can be covered by at most k coins.) Inaba's theorem shows that $10 \leq k$, and trivially there is an upper bound of k; if we put sufficiently many points on a fine lattice, disjoint coins cannot cover all of them (Figure 1). This problem spread over the puzzle society in 2010 (at the 9th Gathering 4 Gardner). Peter Winkler took up this problem in his column [5], and he gave a configuration of 60 points that cannot be covered by disjoint coins. Moreover, Peter Winkler improved the lower bound from 10 to 12 [6, 7]. That is, $12 \leq k \leq 59$. Recently, Veit Elser improved the upper bound to 54 in 2011 [2]. In this paper, we further improve the upper bound of k to 52. That is, we give a configuration of 53 points that cannot be covered by disjoint coins. The main theorem is summarized as follows.

Theorem 2 *Let k be the maximum number such that any configuration of k points can be covered by disjoint coins. Then $12 \leq k \leq 52$.*

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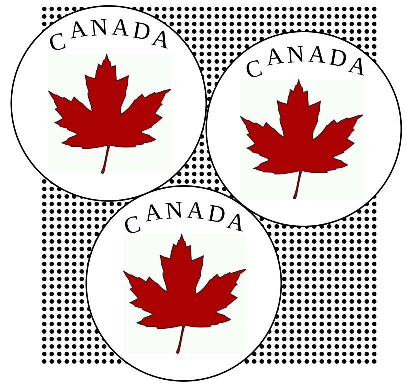

Figure 1: Points cannot be covered by disjoint coins

Hereafter, we assume that each coin is a unit disk of radius 1. To simplify the argument, each unit disk is an open disk. That is, a point on the edge of a unit disk is not covered by the disk. (Using the perturbation technique, our results can be applied to closed disks.) Let L_3, L_4, and L_6 be a triangular, square, and hexagonal lattice, respectively. The *size* of a lattice is defined by the shortest distance between any pair of two points in L_i for $i = 3, 4, 6$ (Figure 2). We sometimes abuse L_i as a set of lattice points for $i = 3, 4, 6$. Our construction of the point configuration consists of two phases.

2 Configuration of the points in a circle

We first consider point configurations in a large circle. We denote by x a circle of radius $r = 2\sqrt{3}/3 - 1 = 0.1547\ldots$. For the circle x, we have the following lemma:

Lemma 3 *Let C_1 and C_2 be disjoint two unit disks. We suppose that a circle x circumscribes both of C_1 and C_2. Then we cannot arrange any unit disk C_3 with $C_3 \cap x \neq \emptyset$ that is disjoint from C_1 and C_2.*

Proof. Since $r = 2\sqrt{3}/3 - 1$, when C_1, C_2, C_3 touch with each other, the circle x also touches all of them (Figure 3). Since C_1, C_2, C_3 are disjoint, the lemma follows immediately. $\square$

Using the circle x of radius r, we give the key idea of our point configuration:

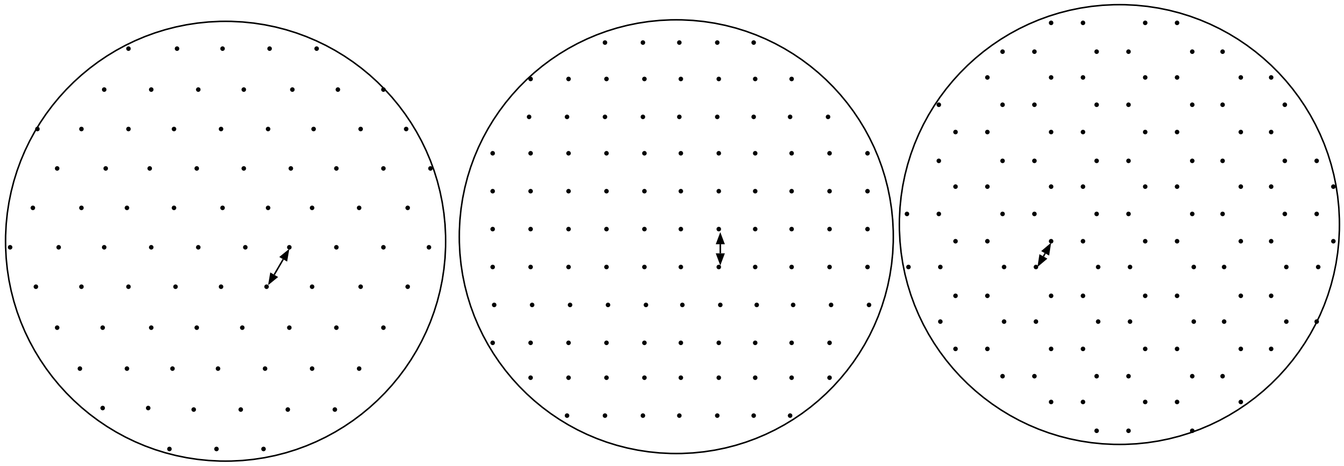

Figure 2: Triangular lattice, square lattice, and hexagonal lattice. Each size is given by the length of the arrow.

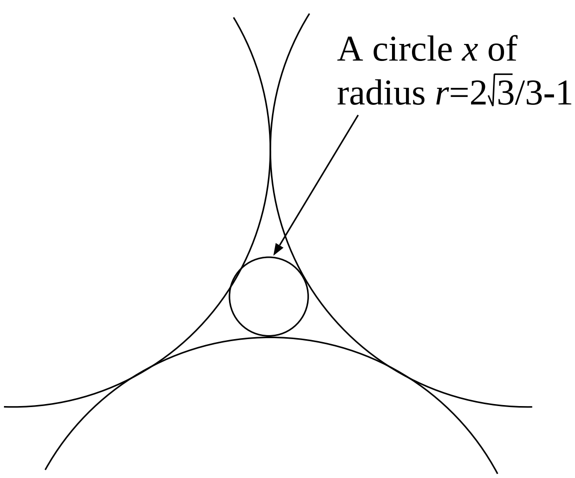

Figure 3: The circle x in the space surrounded by three unit disks

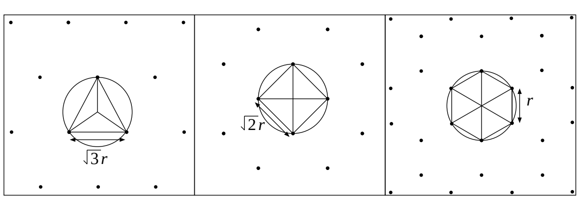

Figure 4: The size of each lattice

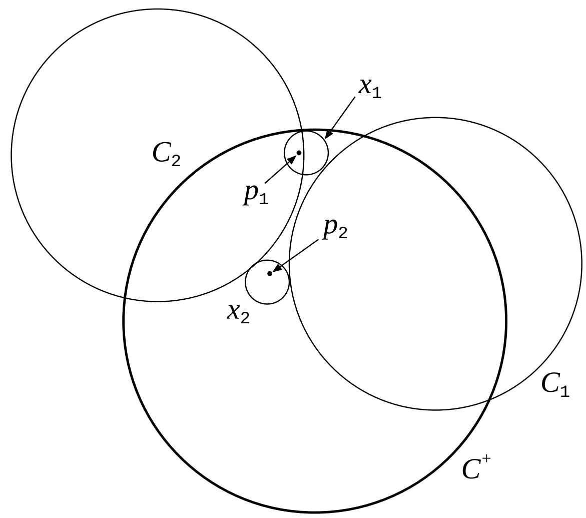

Figure 5: Proof of Lemma 4

Lemma 4 *Let C^+ be a disk of radius $1+2r$. For $i = 3, 4, 6$, let L_i be the lattice of size $\sqrt{3}r$, $\sqrt{2}r$, and r, respectively (Figure 4). (That is, we make x the largest empty circle of each L_i.) Then any point configuration in $L_i \cap C^+$ cannot be covered by disjoint unit disks.*

Proof. We first observe that when we put a closed disk x' of radius r in $L_i \cap C^+$, x' should contain at least one point in $L_i \cap C^+$ because of the size of L_i.

Now in order to derive a contradiction, we assume that all the points in $L_i \cap C^+$ are covered by disjoint unit disks $C_1, C_2, \ldots$. Without loss of generality, $C_1 \cap C^+$ contains the largest number of points in C^+ among $C_i \cap C^+$ (Figure 5). Then we can put a circle x_1 of radius r in $C^+ \setminus C_1$ such that x_1 inscribes C^+ and circumscribes C_1. Then, by the observation, x_1 contains at least one

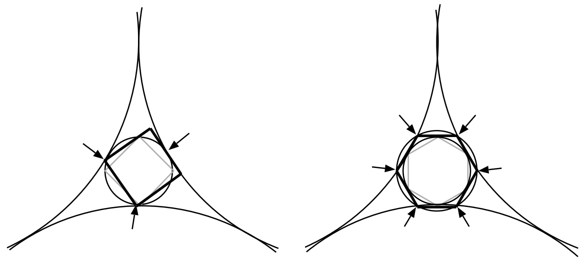

Figure 6: Enlargement of the lattices L_4 and L_6

point p_1 in $L_i \cap C^+$. By the assumption, there is a disk C_2 covering the point p_1 in x_1. Then we can put again a circle x_2 of radius r in $C^+ \backslash C_1$ such that x_2 circumscribes C_1 and C_2, and x_2 contains a point p_2 in $L_i \cap C^+$. (Note that x_1 and x_2 may overlap.) Then, by Lemma 3, we cannot cover p_2 by the other unit disks $C_3, \ldots$. This is a contradiction. Thus the lemma follows. □

By Lemma 4, we can use $L_3 \cap C^+$ of size $\sqrt{3}r$, $L_4 \cap C^+$ of size $\sqrt{2}r$, and $L_6 \cap C^+$ of size r, where $r = 2\sqrt{3}/3 - 1$, as point configurations that give upper bounds of k, respectively. Among them, the upper bound $k < 82$ given by $L_3 \cap C^+$ is much better than the others (the leftmost one in Figure 2). For L_4 and L_6, we can slightly enlarge the size of the lattices than that of Lemma 4 with careful analyses. For L_4, when four points around on x in Figure 4, at most one point on x touches surrounding unit disks. Hence we can enlarge L_4 until at most three points of the square touch surrounding unit disks (Figure 6). (More precisely, we can enlarge to the minimum square of all the squares of which three points of it touch the surrounding disks.) For L_6, we can enlarge L_6 in Figure 4 until all of 6 points are on surrounding unit disks as in Figure 6. However, these enlargements cannot catch up with the case of L_3 at all. Even using the enlargement technique, our best achievements of the cases of L_4 and L_6 are $k < 102$ and $k < 119$, respectively. (The point configurations after enlargements are given in Figure 2.) Hence we omit the details of these enlargements.

3 Improvement of the point configuration

Hereafter, we fix the lattice L_3 of size $\sqrt{3}r$. Carefully checking the proof of Lemma 4, we can see that C^+ is redundant. We first cut off the top and the bottom of C^+ as in Figure 7. More precisely, the lines AB and EF are straight line segments in parallel, and the distance between AB and the center of C^+ is equal to the distance between EF and the center of C^+. The distance between AB and EF is $1 + 3r$. The curves HA, BC, DE, and FG are arcs of the circles of radius r. The curves CD and GH are arcs of the circle C^+

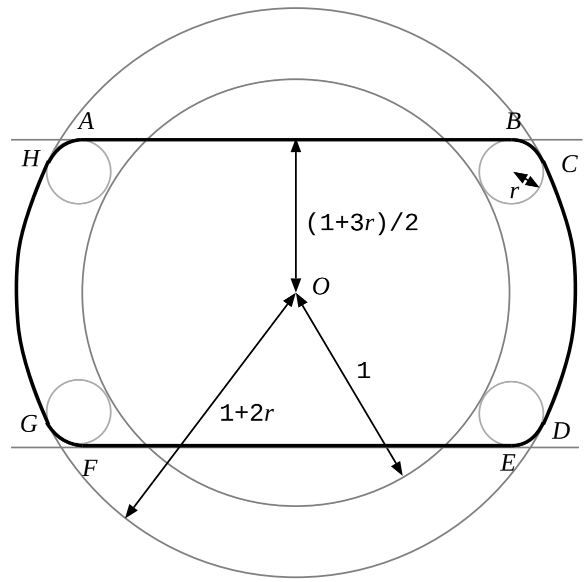

Figure 7: The oval-like form Θ

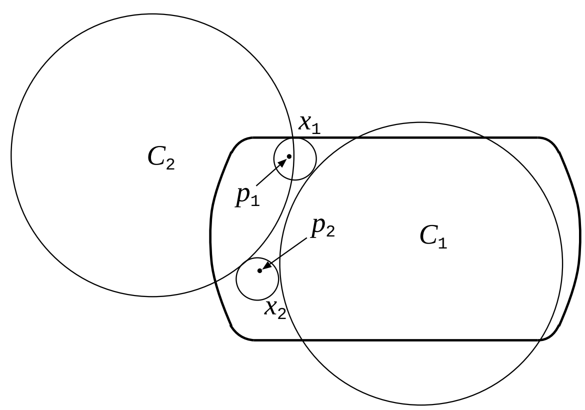

Figure 8: Proof of Lemma 5

of radius $1 + 2r$. Let Θ be the closed area surrounded by the resulting oval-like form $ABCDEFGH$. We now refine Lemma 4:

Lemma 5 *Let Θ be the closed area given by the oval in Figure 7. Let L_3 be the lattice of size $\sqrt{3}r$. Then any point configuration in $L_3 \cap \Theta$ cannot be covered by disjoint unit disks.*

Proof. In order to derive a contradiction, we assume that all points in $L_3 \cap \Theta$ are covered by disjoint unit disks $C_1, C_2, \ldots$. Without loss of generality, $C_1 \cap \Theta$ contains the largest number of points in $L_3 \cap \Theta$ among $C_i \cap \Theta$. Then we can put a circle x_1 of radius r in $\Theta \setminus C_1$ such that x_1 inscribes Θ and circumscribes C_1 (Figure 8). Then x_1 contains at least one point p_1 in $L_3 \cap \Theta$. By the assumption, there is a disk C_2 covering

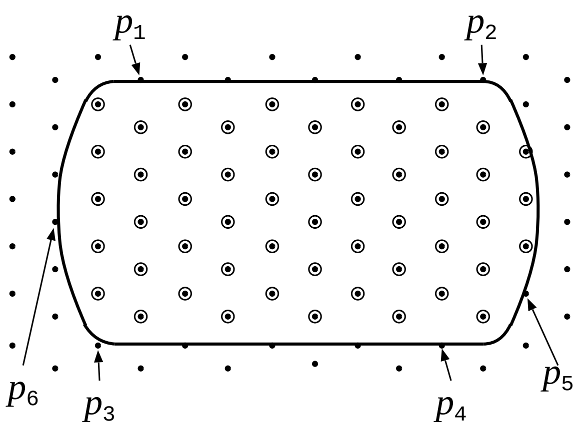

Figure 9: A point configuration in Θ; the circled points are in Θ.

the point p_1. Then we can put again a circle x_2 of radius r in $\Theta \setminus C_1$ such that x_2 circumscribes C_1 and C_2, and x_2 contains a point p_2 in $L_3 \cap \Theta$. Then, by Lemma 3, we cannot put any unit disk that covers p_2. This is a contradiction. Hence the lemma follows. □

Now we minimize the number of points in $L_3 \cap \Theta$, where L_3 has size $\sqrt{3}r$. Our best achievement is given in Figure 9. In this point configuration, we have two criteria for the points $p_1, p_2, \ldots, p_6$ in Figure 9.

1. The line ℓ_1 joining p_1 and p_2 and the line ℓ_2 joining p_3 and p_4 have enough distance to put Θ between them; the distance between ℓ_1 and ℓ_2 is equal to $5.5\sqrt{3}r = 5.5\sqrt{3}(2\sqrt{3}/3 - 1) = 5.5(2 - \sqrt{3}) = 1.4737\ldots$ On the other hand, the corresponding width of Θ is equal to $1 + 3r = 1 + 3(2\sqrt{3}/3 - 1) = 2\sqrt{3} - 2 = 1.4641\ldots$. Hence we can put Θ between ℓ_1 and ℓ_2 such that all the points on ℓ_1 or ℓ_2 are outside of Θ.

2. In Figure 9, the closest points on the right and left sides of Θ are p_5 and p_6, respectively. We show that we can put Θ between them. To simplify the argument, we assume that we put Θ on the line ℓ_2 (joining p_3 and p_4) as in Figure 9, and we take the coordinate with the center $O = (0, 0)$ of the Θ. Let $p_5 = (x_5, y_5)$ and $p_6 = (x_6, y_6)$. Then we have $p_5 = (x_5, -7\sqrt{3}r/4)$, $p_6 = (x_6, -\sqrt{3}r/4)$, and $|x_5 - x_6| = 33r/2 = 11\sqrt{3} - 33/2 = 2.5525\ldots$. Let p'_5 be the point on the edge of Θ such that p'_5 has the same height of p_5 (and closest one of two such points). Let p'_6 be the point on the edge of Θ defined similarly for p_6. That is, we can let $p'_5 = (x'_5, -7\sqrt{3}r/4)$, and $p'_6 = (x'_6, -\sqrt{3}r/4)$. Since $x'^2_i + y'^2_i = (1 + 2r)^2$ for $i = 5, 6$, we can obtain $|x'_5 - x'_6| = \sqrt{115/(4\sqrt{3}) - 725/48 + \sqrt{283/48 - 29/(4\sqrt{3})}} = 2.5302\ldots$. Therefore, we can put Θ between p_5 and p_6 such that they are outside of Θ.

Based on these criteria, we can put Θ as in Figure 9, and the only circled points are in Θ. The number of the circled points is 53, and that concludes the proof of Theorem 2.

4 Concluding remarks

We give an upper bound of 52 for the maximum number k such that any configuration of k points can be covered by disjoint coins. In the oval Θ, it is essentially required that the radius of the largest empty circle is bounded by $r = 2\sqrt{3}/3 - 1$. Hence some computational power may improve the upper bound. But smart proofs seem to be better; recently, Aloupis developed another technique, and gave a better upper bound [1]. Applying his technique to the point configuration in Figure 9, it seems that we can remove a few more points. Our idea is based on the uniform point configurations. The upper bound based on some nonuniform point configurations would be interesting.

We still have a big gap between 12 and 52. Improvement of the lower bound is also interesting. In the appendix, we give the proof of the lower bound 10 by the probabilistic method. Indeed, the proof states a stronger result: any configuration of 10 points can be covered by the sheet in Figure 10. That is, the arrangement of the coins is fixed. Moreover, the bound given by the probabilistic method does not seem to be tight. Hence the gap between the lower bound and the real value seems to be larger than the gap between the upper bound and the real value.

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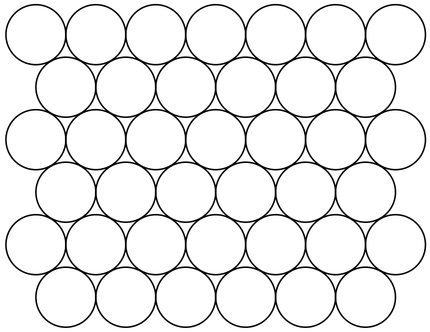

Figure 10: A sheet of infinitely many coins

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A Proof of Inaba's theorem by the probabilistic method

Let P be any configuration of 10 points $p_1, p_2, \ldots, p_{10}$. We put randomly a sheet of infinitely many coins arranged like Figure 10 on P. For $i = 1, 2, \ldots 10$, let A_i be the event that the point p_i is covered by a coin. Then, $\Pr\{A_i\} = (\sqrt{3} - \pi/2)/\sqrt{3} > 0.093$ by a simple calculation of ratios of areas of coins and the background. Hence the probability that all points are covered is given as follows:

$$
\begin{aligned}
&\Pr\{A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_{10}\} \\
&= \quad 1 - (\Pr\{\overline{A_1 \wedge A_2 \wedge A_3 \wedge \cdots \wedge A_{10}}\}) \\
&= \quad 1 - (\Pr\{\overline{A_1} \vee \overline{A_2} \vee \overline{A_3} \vee \cdots \vee \overline{A_{10}}\}) \\
&\geq \quad 1 - (\Pr\{\overline{A_1}\} + \Pr\{\overline{A_2}\} + \Pr\{\overline{A_3}\} + \cdots + \Pr\{\overline{A_{10}}\} \\
&> \quad 1 - 10 \cdot 0.093 = 0.07 > 0.
\end{aligned}
$$

Since the all points are covered with positive probability, there exists a way to put the sheet to cover all the points.