Weak Matching Points with Triangles

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Abstract

In this paper, we study the weak point matching problem for a given set of n points and a class of equilateral triangles. The problem is to find the maximum cardinality matching of the points using equilateral triangles such that each triangle contains exactly two points and each point lies at most in one triangle. Under the non-degeneracy assumption, we present an $O(n^{3/2})$ time algorithm using the TD-Delaunay graph and a graph matching algorithm. Also, we show that the lower bound for the number of matched points is $\lfloor 2n/3 \rfloor$ which is optimal in the worst case.

1 Introduction

The point matching problem is a challenging problem in computational geometry and graph theory and has many applications in geometric shape matchings and computational biology [3]. The problem of point matching with planar geometric objects, recently studied in [1], is a special variant of point matching problems. Given a set P of points in the plane and a class C of $2D$ geometric objects, the problem is to find a set of C-type objects, called *C-matching* of P, in which each object contains exactly two points of P and each point lies in at most one object. The problem is a generalization of geometric graph matching where the objects are segments. Alternatively, what we refer to as objects can be circles, squares or rectangles as well.

Assume that the number of points is even. A C-matching is called *perfect* if all points in P are covered, and it is *strong* if the matched geometric C-objects are non-overlapping. In addition, the matching is called *weak* if we do not know whether it is strong [4]. Álbrego et al. studied properties of C-matching problem for two classes of circles and isothetic squares in perfect and strong matching [1]. Assuming the class of objects to be circles, they proved some bounds for the cardinality of matching in strong and/or perfect matching. The weak perfect matching problem for line segments was studied by Rendl and Woeginger [12]. They proposed an $O(n \log n)$ time algorithm for orthogonal segments, where n is the number of points in P. They proved that the problem is NP-complete if the segments are not allowed to cross. Aloupis et al. investigated matching problems for non-crossing objects [3]. They showed that the problem is NP-complete for lines and line segments in general, but polynomial-time when segments form a convex polygon. Also, a bichromatic version of the problem and a non-intersecting constraint have been studied for strong matching when the objects are segments by Dumitrescu and Steiger [8] and Kaneko and Kano [9], respectively.

Álbrego et al. studied the matching problem for circles and squares [1], [2]. Under the non-degeneracy assumption, they showed that there always exists a weak perfect matching for the class of axis-aligned square objects, and proposed a $2\lceil n/5 \rceil$ bound for the cardinality of matching for the strong one. They presented a $2\lceil (n-1)/8 \rceil$ bound for circles, as well. The classes of rectangles and squares have been studied by Bereg et al. [4]. Without the general position assumption, they proposed an $O(n \log n)$ optimal time algorithm for squares in the weak matching realization and an $O(n^2 \log n)$ time algorithm for the strong one. Also, they showed that a weak rectangle matching of maximum cardinality can be computed in $O(\beta n^{1.5})$ time, where β is the minimum of the number of different x-coordinates and the number of different y-coordinates in P. In addition, they proved that there exists an optimal worst case $\lfloor 2n/3 \rfloor$ cardinality of matching for axis-aligned rectangles in the strong matching and proved that the problem of determining whether a given set of points has a perfect strong matching is NP-hard for the class of squares.

In this paper, we study the problem of weak point matching using *equilateral triangles* with a horizontal base which lies below its non-adjacent vertex. We denote this problem of Weak Triangle Matching by *WTM*. The approach that we present is also applicable for homothets of any fixed triangle, by applying a shear transformation. To solve the problem, we use a *shrinkability* property [2] and reduce WTM to a graph matching problem. When two points of P named p and q are matched in a solution to a matching problem, a C-type object contains exactly p and q. Thus, the object can be shrunk such that p and q lie on its boundary. This property is called "shrinkability" of geometry object matching. Having this property, we reduce the problem of

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matching with geometric objects to a graph matching problem. The corresponding graph for the WTM problem is similar to a Θ_k-*graph* which has been used in the geometric spanner context [11]. Indeed, the graph is a special form of a 2-spanner, Θ_6-graph, introduced by Bonichon et al. [6] and called *half*-Θ_6-*graph* [7]. They proved that the *half*-Θ_6-graph is the same as triangular-distance Delaunay graph and can be computed in optimal $O(n \log n)$ time for a set of n points in the plane.

In the next section, we propose an $O(n^{3/2})$ time algorithm for finding the maximum-cardinality matching for the WTM problem. Later in section 3, we will show that the number of matching points with our proposed algorithm will at least be $\lfloor 2n/3 \rfloor$ points for every given point set, which is optimal in the worst case.

2 Weak Point Matching With Equilateral Triangles

The problem of matching with geometric objects has been studied for classes of segments, circles, squares and rectangles. It would be interesting to study the same for convex polygons as well. In this paper, we study equilateral triangles. For the class of arbitrary triangles, the problem will be trivial, because each segment can be assumed to be a triangle with a height sufficiently small. We consider the x-axis aligned equilateral triangles. They are equilateral triangles, one of the edges of which is parallel to the x-axis. We assume that the triangle is located above this edge.

For both strong and weak versions of the problem, there are counterexamples that show a perfect triangle matching does not always exist. But we show in this section that there is an $O(n^{3/2})$ time algorithm which can compute a weak triangle matching of *maximum cardinality* for a set of n points.

For a given set of points $P = \{p_1, p_2, \ldots, p_n\}$, the problem of weak triangle matching called WTM is to find a set of x-axis aligned triangles such that each triangle includes exactly two points of P. Fig. 1 shows two solutions of the WTM problem for a set of eight points.

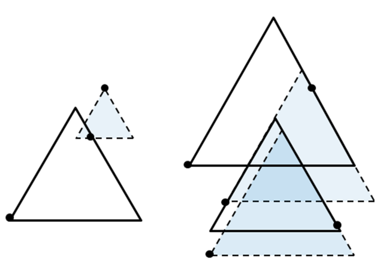

Figure 1: An example of the WTM problem and two distinct solutions for it (dashed triangles and solid triangles).

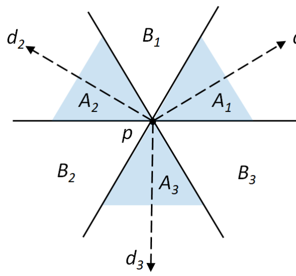

Figure 2: The three directions d_1, d_2 and d_3 and the cones in the covering of a point.

Throughout this paper, we consider three axes d_1, d_2 and d_3 which have angles of $\pi/6$, $5\pi/6$, and $9\pi/6$ with x-axis, respectively. We assume that the points of the set P are in general position, which as we define it, means that there are no two points with the same coordinates in the directions d_1, d_2 or d_3. Also, we denote the orthogonal projection of a point p onto d_i by $d_i(p)$, for i=1, 2 and 3. For a point p, we partition the plane into six regular cones with the apex p. see Fig. 2. The three odd cones with their bisectors being d_1, d_2 and d_3 will be denoted A_1, A_2 and A_3 respectively; the remaining three will be called B_1, B_2 and B_3. We say that the point q is in the covering of p in the direction d_i, if it lies in A_i, for i=1, 2 and 3.

Let T be an axis-aligned equilateral triangle including p and q. We can shrink T to find a smaller such triangle so that p and q lie on its boundary. In addition, for the smallest covering x-axis aligned equilateral triangle at least p or q lies on one of its vertices. We denote such a triangle by $T(p, q)$. Without loss of generality, we assume that each triangle which contributes to WTM has a point on one of its vertices and the other point is on its boundary. Also, for two points p and q in P, we say that $T(p, q)$ is a *candidate triangle* for the weak triangle matching problem if it contains no other points of P. Letting p be a vertex of a candidate triangle, the other point should be in the covering of p. With regard to the general position assumption, we have the following observation.

Observation 1 *Any point p in P can be a vertex of at most three candidate triangles.*

To solve the WTM problem, we define a geometric graph and reduce the problem to a graph matching problem. To this end, we construct the geometric graph $G(P)$ for a point set P. Vertices of $G(P)$ are exactly the point set P, and there is an edge between two vertices p and q if and only if $T(p, q)$ is a candidate triangle. Fig. 3 displays a point set and its corresponding geometric graph.

To compute the geometric graph, $G(P)$, we can use the algorithm of Θ_k-graphs for k=6 [11]. This type

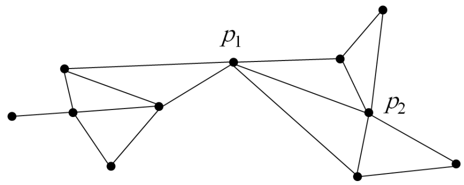

Figure 3: The geometric graph in the WTM problem for a set of points.

of graphs are the linear approximation of complete Euclidean graphs. Chew showed that the Delaunay triangulation using triangle distance function (called TD-Delaunay graph) is a 2-spanner graph [7]. To construct the TD-Delaunay graph it is sufficient to replace the empty equilateral triangle with the circle in the empty circle test in constructing the standard Euclidean Delaunay triangulation. Also, it is proved that the size of the TD-Delaunay graph is linear and can be computed using the sweep line approach in $O(n \log n)$ time for a set of n points. The final result in this context was presented by Bonichon et al. [6]. They introduced a specific subgraph of Θ_6-graph, called the *half*-Θ_6-graph, and proved that it is equal to the TD-Delaunay graph. Based on the mentioned concepts, we can conclude the following result.

Proposition 1 *For a given set P of n points in the plane, the geometric graph $G(P)$ is a connected graph with $O(n)$ edges and can be computed in $O(n \log n)$ time.*

Since an edge in $G(P)$ corresponds with a candidate triangle in P, solving the problem in P is equal to finding the maximum graph matching in $G(P)$. The maximum graph matching for a graph $G = (V, E)$ can be solved using Micali and Vazirani's algorithm in $O(|V|\sqrt{|E|})$ time [10]. Taking into account the linear size of $G(P)$, we conclude this section with the following theorem:

Theorem 2 *For a set of n points in the plane, the maximum cardinality weak point matching with x-axis aligned equilateral triangles can be solved in $O(n^{3/2})$ time and $O(n)$ space.*

3 Lower Bound for the number of matched points for the WTM

In the previous section, we showed that there is an algorithm that finds a maximum cardinality matching for a given point set. In this section, we show that the weak triangle matching for the points in general position always covers at least $\lfloor 2n/3 \rfloor$ points. If the points are not in general position, the worst case is the one in which each point has the same coordinate as another point, in direction d_1, d_2 or d_3 as illustrated in Fig. 4.

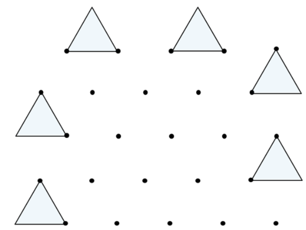

Figure 4: An example for a set of points which are not in general position.

In this case, only the extreme points can be matched. Without the general position assumption, the lower bound for the number of the points which can be matched in an arbitrary point set, P, with the cardinality of n is $O(\sqrt{n})$. If we assume that the points are in general position, the problem of finding the lower bound for the number of matched points with WTM becomes interesting. The following lemmas present some properties of the corresponding graph to find a lower bound.

Lemma 3 *For each two vertices p and q in $G(P)$, there are vertices $r_1, r_2, \ldots, r_k$ ($k \geq 0$) inside $T(p, q)$ such that, the path $pr_1r_2 \ldots r_kq$ is between p and q and each r_i, ($1 \leq i \leq k$) lies in $T(u, v)$ where u and v are the adjacent vertices of r_i on the path $pr_1r_2 \ldots r_kq$.*

Proof. If $T(p, q)$ is a candidate triangle, there is an edge between p and q. So, the lemma holds for k=0. Otherwise, there is a vertex inside $T(p, q)$, e.g. r_1, which $T(p, r_1)$ is a candidate triangle and there is an edge between p and r_1. For the vertices q and r_1, if $T(r_1, q)$ is a candidate triangle, there is an edge between them. So, the lemma holds for k=1. Otherwise, similarly there is a vertex inside $T(r_1, q)$, e.g. r_2, which $T(r_1, r_2)$ is a candidate triangle and there is a path between r_2 and q. Consequently, the path $pr_1r_2 \ldots r_kq$ lies inside $T(p, q)$ and each r_i, ($1 \leq i \leq k$) lies in $T(u, v)$, where u and v are the adjacent vertices of r_i on the path. □

Lemma 4 *For an arbitrary point, q, consider the six mentioned cones, A_i and B_i, for i=1, 2, 3. If there are two points, p_1 and p_2, such that q lies inside $T(p_1, p_2)$, then one of them, e.g. p_1, cannot be in the covering of q and the other point, p_2, cannot be in the cone containing p_1 and its two adjacent cones.*

Proof. If both two vertices, p_1 and p_2, are in the covering of q, then q cannot be inside $T(p_1, p_2)$, because there exists a line that separates q and $T(p_1, p_2)$. For example, if p_1 lies in A_1 and p_2 lies in A_2, $T(p_1, p_2)$ completely lies above the horizontal line that passes through q. So, suppose that p_1 is not in the covering of q and lies in one of B_i cones, e.g. B_1. If q lies inside $T(p_1, p_2)$, then $d_3(p_2) > d_3(q)$ which implies that p_2 cannot be in B_1 or in its adjacent cones, A_1 and A_2. □

Let $C(q)$ be the number of connected components which are created by removing a vertex q from $G(P)$. We will have the following lemmas.

Lemma 5 *For any vertex q in the corresponding graph of the point set P, $G(P)$, $C(q) \leq 3$.*

Proof. Consider the point q and its six mentioned cones. For contradiction, assume that there are at least four components after removing q, so there is a vertex in each component, e.g. p_1, p_2, p_3 and p_4, which connect to q by an edge. See Fig. 5. According to lemma 4, for two points p_i and p_j, for $1 \leq i, j \leq 4$, if q is inside $T(p_i, p_j)$, there is at least one of the cones, A_1, A_2 or A_3 between p_i and p_j, otherwise, there is a path between them which does not pass through q. In this case, there are four vertices lying in 6 regions. So, there are at least two vertices which are in the same or two adjacent cones. This means that by removing q, there is a path between at least two vertices of p_i which does not contain q. It implies that these two vertices which are p_1 and p_4 in Fig. 5, cannot be in two disjoint components, after removing q, which would be a contradiction. □

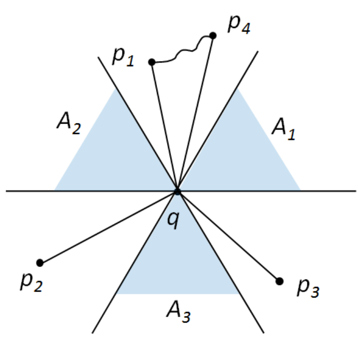

Figure 5: The vertices adjacent to q cannot be in more than three components.

Lemma 6 *Suppose that the vertices $p_1, p_2, \ldots, p_{i-1}$ have been removed from $G(P)$, and $G'(P)$ be the resulted graph. If by removing a vertex, e.g. p_i from $G'(P)$, more than two connected components are added, then there would be two vertices r and s connected to p_i, such that $T(r, s)$ contains some vertex like q where $q \in \{p_1, p_2, \ldots, p_{i-1}\}$ and $C(q) < 3$ but $T(r, s)$ does not contain p_i.*

Proof. According to lemma 5, $C(p_i) \leq 3$. So, each two vertices adjacent to p_i which are in two disjoint components by removing p_i from $G(P)$, the vertex p_i is inside the triangle of them. So, it is expected that removing p_i from $G'(P)$ adds two connected components. Unless, there are two vertices adjacent p_i like r and s such that $T(r, s)$ does not contain p_i, furthermore, by removing p_i from $G(P)$, the vertices r and s are in the same component, while by removing $p_1, p_2, \ldots, p_{i-1}, p_i$, vertices r and s are in two disjoint components. See Fig. 6. It means that in $G'(P)$ there is no edge between r and s. According to lemma 3, there is a path with length of more than one between r and s, and the vertices on the path are inside $T(r, s)$. This path is disjoint from sp_ir, because r and s are in two disjoint connected components. There should be a vertex on the path like q, which has been deleted before. According to lemma 3, q is inside $T(u, v)$ where u and v are adjacent vertices of q on the path between r and s. The path between u and v passing through p_i, implies that the number of created components by removing q from $G(P)$ cannot be three. So, $C(q) < 3$. □

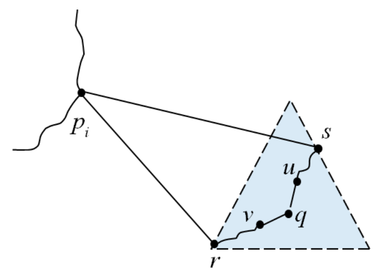

Figure 6: The adjacent vertices of p_i in lemma 6.

Suppose that we want to remove the vertices of a set from the corresponding graph, one-by-one. Note that, the sequence created by the number of added connected components by removing each vertex, varies with the order of removing vertices. For example, in Fig. 3, there are two possible orders for removing the two points, p_1 and p_2. For these two removing orders, p_1, p_2 and p_2, p_1, the sequences of the number of the added connected components are $1, 1$ and $0, 2$, respectively. It is clear that the total number of created connected components is independent of the removing order. The lemmas 5 and 6 show that by removing each vertex, at most two connected components are added, unless there is a vertex which has been removed before and the number of connected components created by removing it from $G(P)$ is less than three. We are going to show that there is a vertex removing order, which guarantees at most two connected components are added by removing each vertex. The following lemma concludes this discussion.

Lemma 7 *By removing the vertices of the set $S = \{p_1, p_2, \ldots, p_k\}$ from $G(P)$, at most $2k + 1$ connected components are created.*

Proof. As we discussed before this lemma, different orders of removing vertices of S generate different sequences of the number of added components. However, the total number of created components is the same. To prove the lemma, we show that there is an order for removing the vertices of S such that at most two connected components are added by removing each vertex. If such an order exits, the number of connected components by removing k vertices, will be at most $2k+1$. Let $Pr(p_i)$ be the priority of removing p_i. For any p_i and p_j in S, if $Pr(p_i) > Pr(p_j)$, we remove p_i before p_j. For each two vertices of $G(P)$, p_i, p_j, if p_i has two adjacent vertices like r and s such that $T(r,s)$ contains p_i, but not p_j, let $Pr(p_i) > Pr(p_j)$. See Fig. 7. Lemma 6 shows that if we follow this priority, in each step, at most two connected components will be added. A problem occurs when there is some vertex like p_t such that $Pr(p_t) > Pr(p_i)$ and $Pr(p_j) > Pr(p_t)$. It means that there is a sequence of vertices which have the cycle of priority. For solving this problem, consider all vertices of S which lie on such priority cycles. First, we remove the vertices which lie on more than one priority cycles. For example in Fig. 7, these vertices are p_t and $p_{t'}$. After removing such vertices, we arbitrarily remove one of the vertices on each of the priority cycles which have no common vertex with any other priority cycles. As these vertices are on a cycle of the graph, by removing them no connected components are added. After removing one of the vertices of the priority cycles, the priority of the other vertices on the priority circles, will become explicit. The other vertices of S will have the arbitrary priority. Since there is an order for removing the vertices of S such that at most two connected components are added by removing each vertex, the number of connected components created by removing k vertices is at most $2k+1$. $\square$

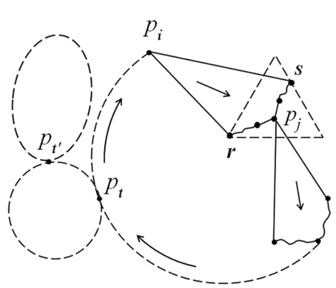

Figure 7: The priority of removing p_i and p_j where $Pr(p_i) < Pr(p_j)$ and the priority cycles.

A basic condition for graphs that have a perfect matching was found by Tutte in 1947. Berge in 1958 observed that it implies a min-max formula for the maximum cardinality $\alpha(G)$ of a matching in a graph G, the *Tutte–Berge* formula. A connected component of a graph is called odd if it has an odd number of vertices. Let $C_o(G)$ denote the number of odd components of G. Then, based on Tutte–Berge formula [5], for each graph $G=(V,E)$,

$$\alpha(G) = \min_{S \subset G}(|V(G)| + |S| - C_o(G-S))$$

Tutte–Berge formula and lemma 7 lead to find a lower bound for the number of matched points in WTM.

Theorem 8 *Maximum cardinality of weak triangle matching for any set of n points in the plane in general position matches at least $\lfloor 2n/3 \rfloor$ points.*

Proof. Let $|S| = k_S$ and G be the corresponding graph of P. According to lemma 7, $C_o(G-S) \leq C_(G-S) \leq 2k_S+1$. Based on the Tutte–Berge formula

$\alpha(G) = \min_{S \subset G}(|V(G)| + |S| - C_o(G-S)) \geq$
$\min_{S \subset G}(n + k_S - 2k_S - 1) = \min_{S \subset G}(n - k_S - 1)$

We consider two following cases:

- $|S| < n/3$

  $M_1 = \min_{S \subset G}(n - k_S - 1) > n - n/3 - 1 > 2n/3 - 1 \Rightarrow M_1 \geq 2n/3$

- $|S| \geq n/3$

  $C_o(G-S) \leq 2k_S + 1 \Rightarrow \forall S, \exists F_S \geq 0,$

  $C_o(G-S) = 2k_S + 1 - F_S,$

  $|S| + C_o(G-S) \leq n \Rightarrow k_S + 2k_S + 1 - F_S \leq n \Rightarrow 3k_S + 1 - F_S \leq n \Rightarrow F_S \geq 3k_S + 1 - n,$

  $M_2 = \min_{S \subset G}(|V(G)| + |S| - C_o(G-S)) =$

  $\min_{S \subset G}(n + k_S - (2k_S + 1 - F_S)) =$

  $\min_{S \subset G}(n - k_S - 1 + F_S) \geq$

  $\min_{S \subset G}(n - k_S - 1 + 3k_S + 1 - n) =$

  $\min_{S \subset G}(2k_S) \geq 2n/3$

Therefore,
$\alpha(G) = min(M_1, M_2) \geq 2n/3 \geq \lfloor 2n/3 \rfloor$

$\square$

Fig. 8 depicts a point set P and its corresponding geometric graph which has $n = 3k$ vertices. As illustrated in the figure, the triangles $T(r_i, s_i)$, $T(r_i, r_j)$ and $T(s_i, s_j)$, for $1 \leq i, j \leq k$, are not candidate triangles. Therefore, the candidate triangles are $T(r_i, m_i)$ and $T(s_i, m_i)$, for $1 \leq i \leq k$, and also, $T(r_i, m_{i-1})$, $T(s_i, m_{i-1})$ and $T(m_i, m_{i-1})$, for $2 \leq i \leq k$. Each edge has an end point at the central vertices, $m_1, m_2, \ldots, m_k$. Clearly, only one of the edges incident to m_i can be in a matching. It shows that the ratio of the points that can be covered by a maximum cardinality weak triangle matching is $2/3$, so, the proposed lower bound is tight.

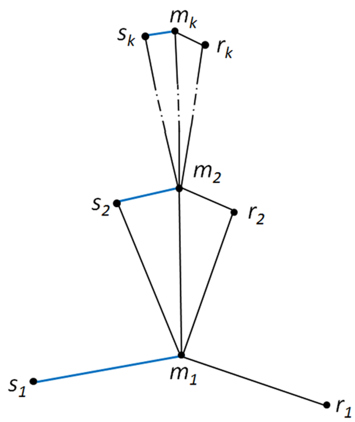

Figure 8: Set of n points which at most $\lfloor 2n/3 \rfloor$ can be matched.

4 Conclusion

The problem of matching points with classes of objects such as circles, squares and rectangles has been recently studied in computational geometry and graph theory. In this paper, we studied the weak point matching for the class of equilateral triangles as an open problem of previous studies. We showed that the maximum cardinality of this kind of matching can be computed using a convex distance function based on equilateral triangles. In addition, we discussed the lower bound of the size of weak triangle matching. We proved that for every point set, at least 2/3 of the points can be matched and we showed that this lower bound is tight. These results are also true for homothets of any fixed triangle. However, the time optimality of the algorithm remains as an open problem. Another future work is to study the strong version of the problem.

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