Development of Curves on Polyhedra via Conical Existence*

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Abstract

We establish that certain classes of simple, closed, polygonal curves on the surface of a convex polyhedron develop in the plane without overlap. Our primary proof technique shows that such curves "live on a cone," and then develops the curves by cutting the cone along a "generator" and flattening the cone in the plane. The conical existence results support a type of source unfolding of the surface of a polyhedron, described elsewhere.

1 Introduction

Nonoverlapping development of curves plays a role in unfolding polyhedra without overlap [2]. Any result on simple (non-self-intersecting) development of curves may help establishing nonoverlapping surface unfoldings. One of the earliest results in this regard is [7], which proved that the left development of a directed, simple, closed convex curve does not self-intersect. The proof used Cauchy's Arm Lemma. Here we extend this result to a wider class of curves without invoking Cauchy's lemma. Our results support a "source unfolding" based on these curves, described in [5].

Development. Let C be a simple, closed, polygonal curve on the surface of a convex polyhedron $\mathcal{P}$. For any point $p \in C$, let $L(p)$ be the total surface angle incident to p at the left side of C, and $R(p)$ the angle to the right side. The *left development* of C with respect to $x \in C$ is an isometric drawing $\overline{C_x}$ of C in the plane, starting from x, such that the angle to the left of $\overline{C_x}$ at every point in the plane is $L(p)$. The *right development* is defined analogously. The left and right developments of a curve are different if C passes through one or more vertices of P. And in general the development depends upon the *cut point* x.

Curve Classes. To describe our results, we introduce a number of different classes of curves on convex polyhedra, which exhibit different behavior with respect to living on a cone. Define a curve C to be *convex* (to the left) if the angle to the left is at most π at every point p: $L(p) \leq \pi$; and say that C is a *convex loop* if this condition holds for all but one exceptional *loop point* x, at which $L(x) > \pi$ is allowed. Analogously, define C to be a *reflex curve* if the angle to one side (we consistently use the right side) is at least π at every point p: $R(p) \geq \pi$; and say that C is a *reflex loop* if this condition holds for all but an exceptional loop point x, at which $R(x) < \pi$.

The loop versions of these curves arise naturally in some contexts. For example, extending a convex path on $\mathcal{P}$ until it self-intersects leads to a convex loop.

Summary of Results.

1. Every convex curve C left-develops to $\overline{C_x}$ without intersection, for every cut point x. This is a new proof of the result in [7].

2. There are convex loops C such that, for some x, the left-development $\overline{C_x}$ self-intersects. However, for every convex loop, there exists a y for which $\overline{C_y}$ left-develops without overlap.

3. Every reflex curve C right-develops to $\overline{C_x}$ without intersection, for every cut point x.

4. Every reflex loop C whose other side is convex right-develops to $\overline{C_x}$ without intersection, for every cut point x.

These results may be combined to reach conclusions about the left- and right-developments of the same curve: Every convex curve C that passes through at most one vertex, both left-develops, and right-develops without overlap, for every cut point x.

Living on a Cone. Our primary proof technique relies on the notion of a curve C "living on a cone," which is based on neighborhoods of C. An open region N_L is a *vertex-free left neighborhood* of C to its left if it includes C as its right boundary, and it contains no vertices of $\mathcal{P}$. In general C will have many vertex-free left neighborhoods, and all will be equivalent for our purposes. We say that C *lives on a cone* to its left if there exists a cone Λ and a neighborhood N_L so that

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$C \cup N_L$ may be embedded isometrically onto Λ, and encloses the cone apex a.

A *cone* is an unbounded developable surface with curvature zero everywhere except at one point, its *apex*, which has total incident surface angle, called the *cone angle*, of at most 2π. Throughout, we will consider a cylinder as a cone whose apex is at infinity with cone angle 0, and a plane as a cone with apex angle 2π. We only care about the intrinsic properties of the cone's surface; its shape in $\mathbb{R}^3$ is not relevant for our purposes. So one could view it as having a circular cross section, although we will often flatten it to the plane.

We should remark that the cone on which a curve C lives has no direct relationship (except in special cases) to the surface that results from extending the faces of $\mathcal{P}$ crossed by C.

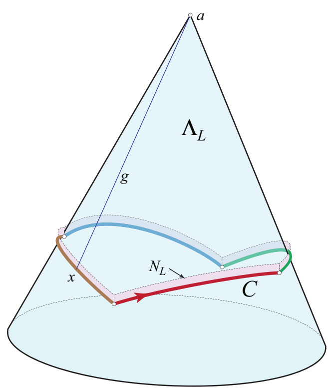

Figure 1: A 4-segment curve C which lives on cone Λ_L to its left. One possible N_L is shown, and a generator $g = ax$ is illustrated.

To say that $C \cup N_L$ embeds isometrically into Λ means that we could cut out $C \cup N_L$ and paste it onto Λ with no wrinkles or tears: the distance between any two points of $C \cup N_L$ on $(C \cup N_L) \cap \mathcal{P}$ is the same as it is on $(C \cup N_L) \cap \Lambda$. See Figure 1. We say that C lives on a cone to its right if $C \cup N_R$ embeds isometrically on the cone, where N_R is a vertex-free right neighborhood of C such that the cone apex a is inside (the image of) C. We will call the cones to the left and right of C, Λ_L and Λ_R respectively. We will see that all four combinatorial possibilities occur: C may not live on a cone to either side, it may live on a cone to one side but not to the other, it may live on different cones to its two sides, or live on the same cone to both sides.

Cone Generators and Visibility. A *generator* of a cone Λ is a half-line starting from the apex a and lying on Λ. A curve C that lives on Λ is *visible* from the apex if every generator meets C at one point. Although it is possible for a curve to live on a cone but not be visible from its apex, when we can establish visibility from the apex, then cutting C at any point $x \in C$ will develop $\overline{C_x}$ without overlap.

2 Preliminary Tools and Lemmas

C partitions $\mathcal{P}$ into two *half-surfaces*. We call the left and right half-surfaces P_L and P_R respectively, or P if the distinction is irrelevant. We view each half-surface as closed, with boundary C.

Curvature. The *curvature* $\omega(p)$ at any point $p \in \mathcal{P}$ is the "angle deficit": 2π minus the sum of the face angles incident to p. The curvature is only nonzero at vertices of $\mathcal{P}$; at each vertex it is positive because $\mathcal{P}$ is convex. The curvature at the apex of a cone is similarly 2π minus the cone angle.

Define a *corner* of curve C to be any point p at which either $L(p) \neq \pi$ or $R(p) \neq \pi$. Let $c_1, c_2, \ldots, c_m$ be the corners of C, which may or may not also be vertices of $\mathcal{P}$. C "turns" at each c_i, and is straight at any non-corner point. Let $\alpha_i = L(c_i)$ be the surface angle to the left side at c_i, and $\beta_i = R(c_i)$ the angle to the right side. Also let $\omega_i = \omega(c_i)$ to simplify notation. We have $\alpha_i + \beta_i + \omega_i = 2\pi$ by the definition of curvature. These definitions will be used to further detail the relationships among the curve classes in Section 5.

The Gauss-Bonnet Theorem. We will employ this theorem in two forms. The first is that the total curvature of $\mathcal{P}$ is 4π: the sum of $\omega(v)$ for all vertices v of $\mathcal{P}$ is 4π. It will be useful to partition the curvature into three pieces. Let $\Omega_L(C) = \Omega_L$ be the total curvature strictly interior to P_L, Ω_R the curvature to the right, and Ω_C the sum of the curvatures on C (which is nonzero only at vertices of $\mathcal{P}$). Then $\Omega_L + \Omega_C + \Omega_R = 4\pi$.

The second form of the Gauss-Bonnet theorem relies on the notion of the "turn" of a curve. Define $\tau_L(c_i) = \tau_i = \pi - \alpha_i$ as the left *turn* of curve C at corner c_i, and let $\tau_L(C) = \tau_L$ be the total (left) turn of C, i.e., the sum of τ_i over all corners of C. Thus a convex curve has nonnegative turn at each corner, and a reflex curve has nonpositive turn at each corner. Then $\tau_L + \Omega_L = 2\pi$, and defining the analogous term to the right of C, $\tau_R + \Omega_R = 2\pi$.

Alexandrov's Gluing Theorem. In our proofs we use Alexandrov's theorem [1, Thm. 1, p. 100] that gluing polygons to form a topological sphere in such a way that at most 2π angle is glued at any point, results in a unique convex polyhedron.

Vertex Merging. We now explain a technique used by Alexandrov, e.g., [1, p. 240]. Consider two vertices v_1 and v_2 of curvatures ω_1 and ω_2 on $\mathcal{P}$, with $\omega_1 + \omega_2 < 2\pi$, and cut $\mathcal{P}$ along a shortest path $\gamma(v_1, v_2)$ joining v_1 to

v_2. Construct a planar triangle $T = \bar{v}'\bar{v}_1\bar{v}_2$ such that its base $\bar{v}_1\bar{v}_2$ has the same length as $\gamma(v_1, v_2)$, and the base angles are equal to $\frac{1}{2}\omega_1$ and respectively $\frac{1}{2}\omega_2$. Glue two copies of T along the corresponding lateral sides, and further glue the two bases of the copies to the two "banks" of the cut of $\mathcal{P}$ along $\gamma(v_1, v_2)$. By Alexandrov's Gluing Theorem, the result is a convex polyhedral surface $\mathcal{P}'$. On $\mathcal{P}'$, the points v_1 and v_2 are no longer vertices because exactly the angle deficit at each has been sutured in; they have been replaced by a new vertex v' of curvature $\omega' = \omega_1 + \omega_2$ (preserving the total curvature). Figure 2(a) illustrates this. Here $\gamma(v_1, v_2) = v_1v_2$ is the top "roof line" of the house-shaped polyhedron $\mathcal{P}$. Because $\omega_1 = \omega_2 = \frac{1}{2}\pi$, T has base angles $\frac{1}{4}\pi$ and apex angle $\frac{1}{2}\pi$. Thus the curvature ω' at v' is π. (Other aspects of this figure will be discussed later.)

Note this vertex-merging procedure only works when $\omega_1 + \omega_2 < 2\pi$; otherwise the angle at the apex $\bar{v}'$ of T would be greater than or equal to π.

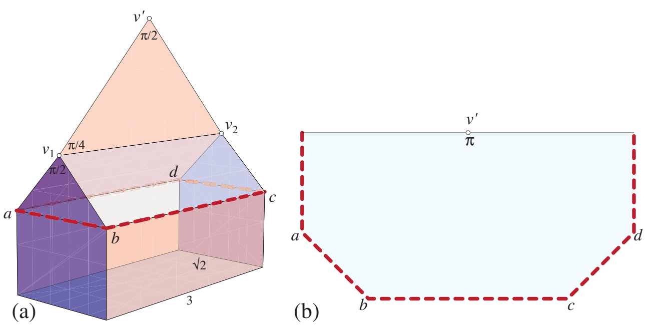

Figure 2: (a) $C = (a, b, c, d)$ is a convex curve with angle $\frac{3}{4}\pi$ to the left at each vertex. The curvature at v_1 and at v_2 is $\frac{1}{2}\pi$. (b) Cutting along the generator from v' through the midpoint of ad and developing C shows that it lives on a cone with apex angle π at v'. (Base of $\mathcal{P}$ is $3 \times \sqrt{2}$.)

Lemma 1 *A curve C that lives on a cone Λ (say, to its left) uniquely determines that cone.*

Proof. *Sketch.* The apex angle of any cone on which C lives must be $\alpha = 2\pi - \Omega_L$, where Ω_L is the total curvature inside and left of C. Imagining rolling out two distinct cones cut along a generator through the same point $x \in C$ leads to isometric unfoldings, showing that the cones are in fact identical. Details are in [5]. □

3 Convex Curves

The lemma below reproves the result from [7].

Lemma 2 *Let C be a convex curve on $\mathcal{P}$, convex to its left. Then C lives on a cone Λ_L to its left side, whose apex a has curvature Ω_L, and so has cone apex angle $2\pi - \Omega_L$. C is visible from the apex a of Λ.*

Proof. *Sketch.* By the Gauss-Bonnet theorem, $\tau_L + \Omega_L = 2\pi$. Because $\tau_L \geq 0$ for a convex curve, we must have $\Omega_L \leq 2\pi$. If $\Omega_L < 2\pi$, we continually merge vertices in P_L until only one remains, at which point P_L is a pyramid, and therefore a cone. If $\Omega_L = 2\pi$, a slight alteration of the proof results in C living on a cylinder. Details are in [5]. □

Example 1. In Figure 2, the two vertices inside C, of curvature $\frac{1}{2}\pi$ each, are merged to one of curvature π, which is then the apex of a cone on which C lives.

Example 2. Figure 3(a) shows an example with three vertices inside C. $\mathcal{P}$ is a doubly covered flat pentagon, and $C = (v_4, v_5, v_4)$ is the closed curve consisting of a repetition of the segment v_4v_5. C has π surface angle at every point to its left, and so is convex. The curvatures at the other vertices are $\omega_1 = \pi$ and $\omega_2 = \omega_3 = \frac{1}{2}\pi$. Thus $\Omega_L = 2\pi$, and the proof of Lemma 2 shows that C lives on a cylinder. Following the proof, merging v_1 and v_2 removes those vertices and creates a new vertex v_{12} of curvature $\frac{3}{2}\pi$; see (b) of the figure. Finally merging v_{12} with v_3 creates a "vertex at infinity" v_{123} of curvature 2π. Thus C lives on a cylinder as claimed. If we first merged v_2 and v_3 to v_{23}, and then v_{23} to v_1, the result is exactly the same, although not obviously so.

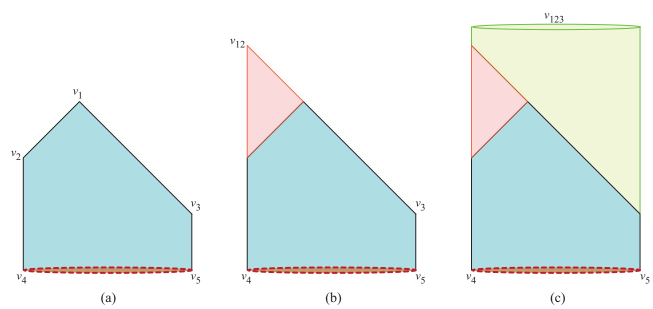

Figure 3: (a) A doubly covered flat pentagon. (b) After merging v_1 and v_2. (c) After merging v_{12} and v_3.

4 Convex Loops

Convex Loops and Cones. We first show that the technique that proved successful for convex curves cannot apply to all convex loops: not every convex loop lives on a cone. Consider the polyhedron $\mathcal{P}$ shown in Figure 4(a), which is a variation on the example from Figure 2(a). Here $C = (a, b, b', x, c', c, d)$ is a convex loop, with loop point x. The cone on which it should live is analogous to Figure 2(b): vertex merging of v_1 and v_2 again produces the cone apex v' whose curvature is π. But C does not "fit" on this cone, as Figure 4(b) shows; the apex $a = v'$ is not inside C.

Overlapping development of convex loop. In light of the preceding negative result, it is perhaps not surpris-

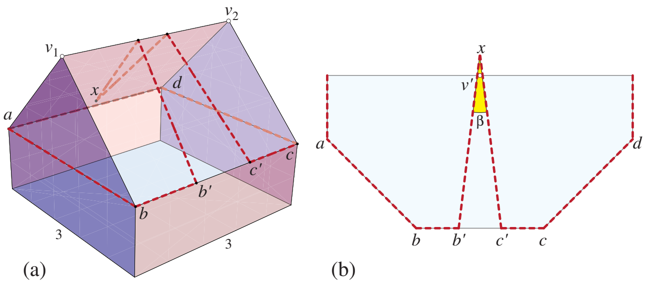

Figure 4: (a) A convex loop C that does not live on a cone. (b) A flattening of the cone on which it should live. (Base of $\mathcal{P}$ is 3×3.)

ing that there are convex loops C and a point $x \in C$ such that $\overline{C_x}$ left-develops with overlap. Indeed Figure 5 shows an example where x is the loop point.

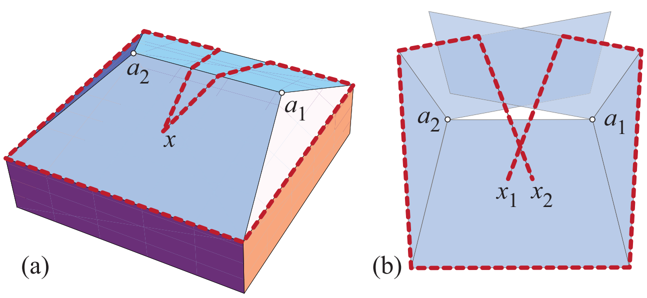

Figure 5: (a) $\mathcal{P}$ with convex loop C. (b) $\overline{C_x}$ when cut at loop point x.

Visibility Points. Despite the negative result illustrated above, we can show that there always exists some cut point y that develops a convex loop without overlap.

Say that $y \in C$ is a *visibility point* for C if for every point $z \in C$ there is a shortest path joining y to z that remains interior to C except at its endpoints. The following proof sketch shows, roughly, that a convex loop C lives on the union of two cones (Case I), or on two cones separated by another region (Case II). This suffices to establish a non-overlapping development. The sketch relies at several points on our work on the star unfolding in [4].

Lemma 3 *Every convex loop C has a visibility point y different from its loop point x, and $\overline{C_y}$ left-develops without overlap.*

Proof. *Sketch.* Let τ_1 and τ_2 be the tangent directions of C at x, and consider $\mu_i = -\tau_i$.

Case I. Assume first there exists a shortest path $\gamma = xy$ from x to some $y \in C$ whose tangent direction at x lies between μ_1 and μ_2; see Figure 6. Then γ splits $P = P_L$ into two convex regions P_i sharing the common boundary point y, and hence (by Lemma 8 in [4]) y "sees" every point in P. Moreover, vertex merging in each P_i produces two cones Λ_i (of apices a_i) with common boundary γ.

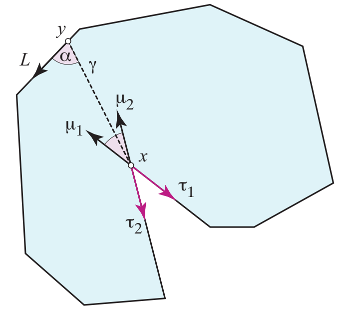

Figure 6: Case 1: $\gamma = xy$ is a shortest path.

Claim 1. Cutting each cone along the generator $a_i y$ unfolds the union of cones without overlappings. Consequently, this develops C without overlap.

Case II. Assume now that Case I does not hold. Then P must contain a "fat digon" D, a concept from [4]. This is a region bounded by two shortest paths from x to some $y \in C$ whose angle at x covers all possible "splitting" γ between μ_1 and μ_2. In this case what remains outside the digon is the union of two convex regions P_i, each visible from y. Moreover, D is itself completely visible from y (see Sec. 4.2 in [4]). Again we perform vertex merging in each P_i to obtain two cones, of apices a_i, which we unfold by cutting along $a_i y$.

We unfold D by the star unfolding with respect to y, and apply Lemma 7 in [4] to establish that the result lies inside some angle (at x).

Claim 2. We can join the unfoldings without overlappings. Consequently, this develops C without overlap. The proof of Claim 2 follows the one for Claim 1, with the additional fact that the star unfolding of the "fat digon" fits inside a circular sector at $\bar{x}$. □

This result on convex loops is best possible in the sense that there are curves C that are convex except at two exceptional points, and for which $\overline{C_x}$ overlaps for every x.

5 Reflex Curves and Reflex Loops

For each corner c_i of a curve C, $\alpha_i + \omega_i + \beta_i = 2\pi$, where α_i and β_i are the left and right angles at c_i respectively, and ω_i is the curvature at c_i. When C is vertex-free, $\omega_i = 0$ at all corners, and the relationships among the curve classes is simple and natural: the other side of a convex curve is reflex, the other side of a reflex curve is convex. The same holds for the loop versions: the other side of a convex loop is a reflex loop (because $\alpha_m \geq \pi$ implies $\beta_m \leq \pi$, where c_m is the loop point), and the other side of a reflex loop is a convex loop. When C

includes vertices, the relationships between the curve classes are more complicated. The other side of a convex curve is reflex only if the curvatures at the vertices on C are small enough so that $\alpha_i + \omega_i \leq \pi$; C would still be convex even if it just included those vertices inside. The same holds for convex loops.

On the other hand, the other side of a reflex curve is always convex, because nonzero vertex curvatures only make the other side more convex. The other side of a reflex loop is a convex loop, and it is a convex curve if the curvature at the loop point c_m is large enough to force $\alpha_m \leq \pi$, i.e., if $\beta_m + \omega_m \geq \pi$.

This latter subclass of reflex loops—those whose other side is convex—especially interest us, because any convex curve that includes at most one vertex is a reflex loop of that type. All our results in this section hold for this class of curves.

Lemma 4 *Let C be a curve that is either reflex (to its right), or a reflex loop which is convex to the other (left) side, with $\beta_m < \pi$ at the loop point c_m. Then C lives on a cone Λ_R to its reflex side, and is visible from its apex a. If $\Omega_R > 2\pi$, then the reflex neighborhood N_R is to the unbounded side of Λ_R, i.e., the apex of Λ_R is left of C; if $\Omega_R < 2\pi$, then N_R is to the bounded side, i.e., the apex of Λ_R is to the right side of C. If $\Omega_R = 2\pi$, $C \cup N_R$ lives on a cylinder.*

Proof. *Sketch.* Because C is convex to its left, we have $\Omega_L \leq 2\pi$. Just as in Lemma 2, merge the vertices strictly in P_L to one vertex a. Let Λ_L be the cone with apex a on which C now lives.

The remainder of the proof alters Λ_L to Λ_R step-by-step with repeated insertions of "curvature triangles" to the left at each corner c_i of C. Each of these triangles is an isosceles triangle of apex angle ω_i, which flattens the surface at c_i without altering $C \cup N_R$. For a detailed proof, see [5] □

Example 3. An example of a reflex loop that satisfies the hypotheses of Lemma 4 is shown in Figure 7(a). Here C has five corners, and is convex to one side at each. C passes through only one vertex of the cuboctahedron $\mathcal{P}$, and so it is reflex at the four non-vertex corners to its other side. Corner c_5 coincides with a vertex of $\mathcal{P}$, which has curvature $\omega_5 = \frac{1}{3}\pi$. Here $\alpha_5 = \beta_5 = \frac{5}{6}\pi$. Because $\beta_5 < \pi$, C is a reflex loop. We have $\Omega_L = \frac{2}{3}\pi$ because C includes two cuboctahedron vertices, u and v in the figure. $\Omega_C = \omega_5 = \frac{1}{3}\pi$. And therefore $\Omega_R = 3\pi$. The apex curvature of Λ_L is $\Omega_L = \frac{2}{3}\pi$, and the apex curvature of Λ_R is π. N_R lives on the unbounded side of this cone, which is shown shaded in Figure 7(b). Note the apex a is left of C, in accord with the lemma.

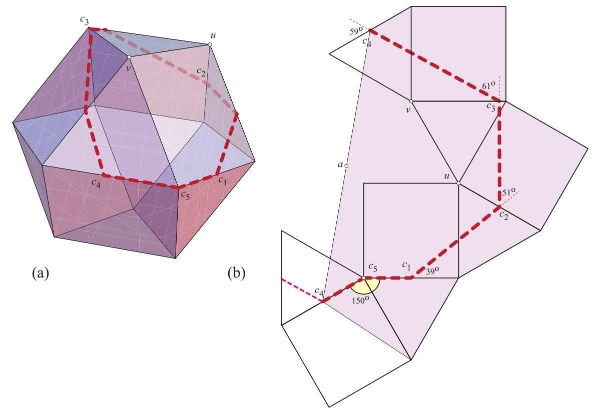

Figure 7: (a) A curve C of five corners passing through one polyhedron vertex. C is convex to one side, and a reflex loop to the other, with loop point c_5, at which $\beta_5 = \frac{5}{6}\pi (= 150°) < \pi$. (b) The cone Λ_R with apex a is shaded.

6 Discussion

We summarize the results claimed in the Introduction in a theorem:

Theorem 5 *On a convex polyhedron, every convex curve left-develops without overlap, and every reflex curve, and reflex loop whose other side is convex, right-develops without overlap, for every cut point. Every convex loop has some cut-point from which it left-develops without overlap.*

Proving that a curve on a convex polyhedron lives on a cone is a powerful technique for establishing that these polyhedron curves develop without overlap. Even when a curve—such as a convex loop—does not live on a cone, still the cone perspective can help prove nonoverlapping development (Lemma 3).

Many questions remain.

Overlapping Developments. It is not the case that every curve that lives on a cone develops without overlap. Here we show that there exist C such that $\overline{C}_x$ is non-simple for every choice of x. We provide one specific example, but it can be generalized.

The cone Λ has apex angle $\alpha = \frac{3}{4}\pi$; it is shown cut open and flattened in two views in Figure 8(a,b). An open curve $C' = (p_1, p_2, p_3, p_4, p_5)$ is drawn on the cone. Directing C' in that order, it turns left by $\frac{3}{4}\pi$ at p_2, p_3, and p_4. From p_5, we loop around the apex a with a segment $S = (p_5, p_6, p_5')$, where p_5' is a point near p_5 (not shown in the figure). Finally, we form a simple closed curve on Λ by then doubling C' at a slight separation (again not illustrated in the figure), so that from p_5 it returns in reverse order along that slightly displaced path to p_1 again. Note that $C = C \cup S \cup C'$ is closed and includes the apex a in its (left) interior.

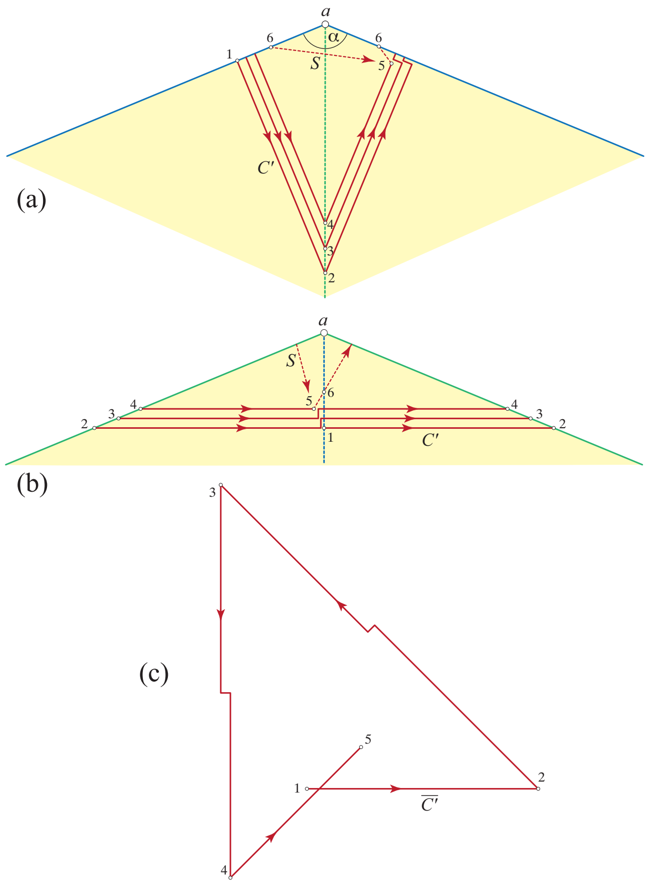

Figure 8: (a) Open curve $C' = (p_1, p_2, p_3, p_4, p_5)$ on cone of angle α, with cone opened. (b) A different opening of the same cone and curve. (c) Development of curve $\overline{C'}$ self-intersects.

Now, let x be any point on C from which we will start the development $\overline{C}_x$. Because C is essentially $C' \cup C'$, x must fall in one or the other copy of C', or at their join at p_1. Regardless of the location of x, at least one of the two copies of C' is unaffected. So $\overline{C}_x$ must include $\overline{C'}$ as a subpath in the plane.

Finally, developing C' reveals that it self-intersects: Figure 8(c). Therefore, $\overline{C}_x$ is not simple for any x. Moreover, it is easy to extend this example to force self-intersection for many values of α and analogous curves. The curve C' was selected only because its development is self-evident.

Slice Curves. There are curves already known to develop without overlap that are not known to live on a cone. One particular class we could not settle are the slice curves. A *slice curve* C is the intersection of $\mathcal{P}$ with a plane. Slice curves in general are not convex. The intersection of $\mathcal{P}$ with a plane is a convex polygon in that plane, but the surface angles of $\mathcal{P}$ to either side along C could be greater or smaller than π at different points. Slice curves were proved to develop without intersection, to either side, in [6], so they are good candidates to live on cones. However, we have not been able to prove that they do.

Convex Loops. Although we have shown that there is some cut point from which every convex loop develops without overlap (Lemma 3), we have not determined all the cut points that enjoy this property.

Cone Curves. Finally, we have not obtained a complete classification of the curves on a cone that develop, for every cut point x, as simple curves in the plane. It would equally interesting to identify the class of curves on cones for which there exists at least one cut-point that leads to simple development. Indeed, the same questions for curves on a sphere are also unresolved [3].

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