Rigidity-Theoretic Constructions of Integral Fary Embeddings

Timothy Sun*

Abstract

Fáry [3] proved that all planar graphs can be drawn in the plane using only straight line segments. Harborth *et al.* [7] ask whether or not there exists such a drawing where all edges have integer lengths, and Geelen *et al.* [4] proved that cubic planar graphs satisfied this conjecture. We re-prove their result using rigidity theory, exhibit other natural families of planar graphs that satisfy this conjecture as immediate corollaries, and also prove a weaker result for all planar graphs in $\mathbb{R}^3$.

1 Introduction

All graphs in this paper are simple and finite. Let $G = (V, E)$ be a planar graph. A *Fary embedding* $\phi_G : V \to \mathbb{R}^2$ of G is an embedding such that the drawing induced by ϕ_G with straight-line edges has no crossing edges. Fáry [3] proved a classic theorem on these embeddings.

Theorem 1 (Fáry [3]) *All planar graphs have a Fary embedding.*

The main idea for the proof was by induction on the number of vertices. A vertex v of degree at most 5 in the interior is deleted from a maximal planar graph G, and v is carefully replaced in a Fary embedding of a triangulation of $G - v$ so that it "sees" all its neighbors. An *integral Fary embedding* is a Fary embedding in which for all adjacent vertices a and b, $||\phi_G(a) - \phi_G(b)||$ is an integer. Harborth *et al.* [7] found integral Fary embeddings for the Platonic graphs, which led them to conjecture the following.

Conjecture 1 *All planar graphs have an integral Fary embedding.*

Previous attacks on this conjecture took the same direction as [3], inductively adding new vertices by using solutions to Diophantine equations. Kemnitz and Harborth [8] outline an idea for a possible proof and a construction for some planar graphs, but their method does not always work. Geelen *et al.* [4] give a partial solution in which they demonstrated that all cubic planar graphs satisfy Conjecture 1. Our method determines when it is possible to perturb an edge length without affecting any other edge lengths and preserving planarity, and we prove Conjecture 1 for planar graphs in which all edges can be perturbed. While that family does not contain all planar graphs, it does contain some well-known families of planar graphs, including all but one of the cubic planar graphs. Unlike the results in [4] and [8], the exact combinatorial characterization for such graphs are known.

*Department of Computer Science, Columbia University, ts2578@columbia.edu

2 Rigidity and Edge Perturbations

A *(d-dimensional) framework* is a pair (G, p) where $p : V(G) \to \mathbb{R}^d$, known as a *configuration*, is a mapping which takes the vertices of G to points in Euclidean d-space. We assume that the image of p does not lie on a hyperplane and that p is injective. A *generic configuration* is one where all vd coordinates are independent over the rationals, and a *generic framework* is a framework with a generic configuration. We say that a framework is *flexible* (in $\mathbb{R}^d$) if there exists a continuous motion of the vertices that preserves edge lengths and is not a Euclidean motion, and that it is *rigid* otherwise. We say that a graph is *(generically) flexible/rigid* if all generic frameworks of that graph are flexible/rigid. While the rigidity of a particular framework is dependent on the choice of configuration, generic rigidity is a property of only the underlying graph.

Let G be a graph. Consider the function f_G that takes a configuration to a vector of all the edge lengths squared. In other words, $f_G : \mathbb{R}^{vd} \to \mathbb{R}^e$ is a function which takes

$$p = (p_1, p_2, ...p_v) \mapsto (..., ||p_i - p_j||^2, ...).$$

The Jacobian $df_G(p)$, called the *rigidity matrix* of (G, p), is an $e \times vd$ matrix where each row corresponds to an edge and encodes the vector between the two vertices incident with the edge. For example, the rigidity matrix of the graph K_3 with the configuration $p_1 = (1, 1), p_2 = (2, -2), p_3 = (0, 3)$ can be written as

$$2 * \begin{array}{c} \\ v_1v_2 \\ v_1v_3 \\ v_2v_3 \end{array} \begin{array}{c} \begin{array}{cccccc} p_{1,1} & p_{1,2} & p_{2,1} & p_{2,2} & p_{3,1} & p_{3,2} \end{array} \\ \begin{pmatrix} -1 & 3 & 1 & -3 & 0 & 0 \\ 1 & -2 & 0 & 0 & -1 & 2 \\ 0 & 0 & 2 & -5 & -2 & 5 \end{pmatrix} \end{array}.$$

By factoring out the 2, the entries of the rigidity matrix can be calculated by simply taking the difference between coordinates. Since all possible edge-length functions can be obtained by some permutation of the edges and swapping the "head" and "tail" of an edge, we refer to df_G as *the* rigidity matrix. Most importantly, we are only interested in the rank of the rigidity matrix, which is not affected by such choices. The derivative of any flex or Euclidean motion on a framework lies in the kernel of the framework's rigidity matrix, so the rank gives an informal notion of how rigid the framework is. In particular, adding an edge which is linearly independent from the other edges reduces the dimension of the space of flexes by one. The rank of the rigidity matrix is dependent on the choice of configuration, so we restrict our attention to a specific subset of all configurations. A configuration p is a *regular point* of f_G if

$$\operatorname{rank} df_G(p) = \max_{q \in \mathbb{R}^{vd}} \operatorname{rank} df_G(q).$$

For $v > d$, let $r(v,d)$ be defined as the quantity $vd - \binom{d+1}{2}$. $vd - \binom{d+1}{2}$ can be informally thought of as the number of degrees of freedom we have in selecting a framework. There are vd coordinates to choose from, but the d-dimensional space of translations and the $\binom{d}{2}$-dimensional space of rotations limit the space of non-congruent frameworks. Asimow and Roth [1] formalized this intuitive notion and proved that at regular points, a framework on more than d vertices is rigid at a regular point if and only if the rank of its rigidity matrix is $r(v,d)$. In the case where v is at most d, the only rigid graphs are the complete graphs.

The theorem by Asimow and Roth [1] roughly states that we need as many edge constraints as degrees of freedom for a framework to be rigid. For example, the graph $K_{3,3}$ is generically rigid in $\mathbb{R}^2$ since it has $9 = r(6,2)$ linearly independent edges. In fact, the only flexible frameworks of $K_{3,3}$ are those whose vertices lie on a conic section, like in Figure 1. However, even when there are exactly $r(v,d)$ edges, the framework may not be rigid. For example, any degree 1 vertex can pivot around its neighbor in the plane. We will present the classic combinatorial characterization of graphs with all independent edges in $\mathbb{R}^2$ in the next section that eliminates such problems.

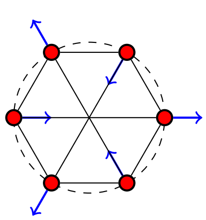

Figure 1: When the configuration is not generic, sometimes the framework is infinitesimally flexible even though the graph is generically rigid.

A *redundant* edge is one whose removal does not decrease the rank of the rigidity matrix. With regards to flexes, removing a redundant edge does not increase the space of flexes. The main technique of this paper is described in the following theorem.

Theorem 2 *Let (G,p) be a framework in $\mathbb{R}^2$ such that p is a Fary embedding and a regular point, and let ab be a non-redundant edge. Then there exists a framework (G,p') such that p' is a Fary embedding and a regular point, $||p'(a) - p'(b)||$ is rational, and all other edge lengths remain fixed.*

Proof. There exists an open neighborhood of configurations N_ϵ around p of Fary embeddings. To see this, we can examine the set of configurations where two fixed edges do not intersect. Since this set is open, the intersection of all such constraints is also open. Furthermore, there exists an open neighborhood of regular points N_η around p. This follows from the fact that there is a maximal rank square submatrix with non-zero determinant and that the determinant is a continuous function on the coordinates.

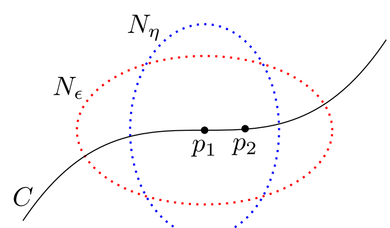

Figure 2: Perturbing an edge. If we remove a non-redundant edge ab of a rigid framework (G,p_1), there will be a non-trivial one-dimensional flex C. By restricting C to $N_\epsilon \cap N_\eta$, we can find a configuration p_2 such that $||p_2(a) - p_2(b)||$ is rational. Then, the edge ab in (G,p_2) has rational length.

Consider the graph formed by removing ab. Since ab is non-redundant, its removal creates a one-dimensional space of flexes. Moving along the flex, the distance between a and b changes, otherwise this flex would be a Euclidean motion. Since the rationals are dense in the reals, we can find a configuration $p' \in N_\epsilon \cap N_\eta$ such that $||p'(a) - p'(b)||$ is rational. Then p' is a regular point ($p' \in N_\eta$) and a Fary embedding ($p' \in N_\epsilon$), $||p'(a) - p'(b)||$ is rational, and all other edge lengths remained constant. □

3 Harborth's Conjecture for $(2,3)$-Sparse Graphs

A graph G is (m,n)-*sparse* if for any subgraph G' with v' vertices and e' edges, $e' \le \max(0, mv' - n)$. Our result in the plane relies on the following characterization.

Theorem 3 (Graver *et al.* [6, Lemma 4.2.1]) *Every edge in (G, p) is non-redundant at a regular point p if and only if G is $(2, 3)$-sparse.*

This result perhaps takes on its most familiar form in Laman [9], who demonstrates that the generically rigid graphs in the plane are exactly the $(2, 3)$-sparse graphs with $2v - 3$ edges, the so-called Laman graphs.

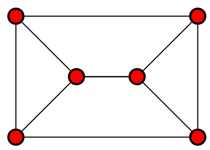

Figure 3: $K_3 \times K_2$, a planar Laman graph.

In the original paper, Laman actually showed only the existence of rigid realizations of Laman graphs, but the density of regular points guarantees that all generic frameworks of Laman graphs are rigid, as well. While the (2,3)-sparseness property does not directly play a part in the proof of the following result, it is useful in characterizing other families of planar graphs that have integral Fary embeddings as corollaries.

Theorem 4 *All planar $(2, 3)$-sparse graphs have integral Fary embeddings.*

Proof. Let G be a $(2, 3)$-sparse graph. We can find a Fary embedding p that is also a regular point by taking a Fary embedding ϕ_G and perturbing it slightly. The regular points are dense in $\mathbb{R}^{vd}$, so this is always possible. By repeatedly applying Theorems 2 and 3, we can perturb all edges to rational lengths by a sequence of configurations $p = p_0 \to p_1 \to \ldots \to p_e$, where each successive term in the sequence is obtained by perturbing another edge in the preceding configuration. Then, p_e is a Fary embedding of G with all rational edge lengths, and scaling appropriately yields an integral Fary embedding. □

An intuitive way to interpret the above result is that if there are few edges and they are evenly spread out among the vertices, it is possible to perturb the lengths of the edges almost however we want. That is, we can choose any rational lengths within some neighborhood of a Fary embedding and obtain an integral Fary embedding by scaling. The construction by Geelen *et al.* [4] can be shown to work on $(2, 3)$-sparse graphs, but their result does not allow for arbitrary choices of rational lengths. On the other hand, Biedl [2] gives an efficient algorithm in the case of 3-connected cubic graphs demonstrating that we can actually choose integer lengths linear in the number of vertices.

Unfortunately, $(2, 3)$-sparseness is far from covering all the planar graphs. When the graph has more than $2v - 3$ edges, we can no longer use this approach since there are redundant edges. Fortunately, this approach is just enough to prove some already-known results. Let G be a *sub-cubic* graph if it has maximal degree 3. We obtain the following results from Theorem 4.

Corollary 5 (Geelen *et al.* [4]) *All sub-cubic planar graphs have integral Fary embeddings.*

Proof. Sub-cubic graphs have at most $\frac{3}{2}v$ edges, so the only sub-cubic graph with more than $2v - 3$ edges is K_4. No connected sub-cubic graph can have K_4 as a proper subgraph because otherwise some vertex would have degree at least 4. Hence, all connected sub-cubic graphs with the exception of K_4 are $(2, 3)$-sparse, so they have an integral Fary embedding by the previous theorem. There are several ways of finding an integral Fary embedding for K_4, the smallest of which can be found using Pythagorean triples as demonstrated in [7]. □

Corollary 6 *Triangle-free planar graphs have integral Fary embeddings.*

Proof. Triangle-free graphs with $v \geq 3$ have at most $2v - 4$ edges, and since a subgraph of a triangle-free graph is also triangle-free, they are $(2, 3)$-sparse. □

Corollary 7 *Bipartite planar graphs have integral Fary embeddings.*

G is a *series-parallel* graph if it is a subgraph of a graph that is constructed from K_2 by adding vertices and attaching them to two adjacent vertices. Wagner [10] proved that a graph is series-parallel if and only if it does not contain K_4 as a minor. Since both $K_{3,3}$ and K_5 have K_4 as a minor, series-parallel graphs are planar. Alternatively, the constructive characterization immediately yields a method of finding integral Fary embeddings.

Corollary 8 *Series-parallel graphs have integral Fary embeddings.*

Proof. Let G be a "maximal" series-parallel graph. As stated above, G can be constructed from adding new vertices and connecting them to adjacent vertices, so G has $2v - 3$ edges. Any subgraph of G is also series-parallel, so G is $(2, 3)$-sparse. □

Corollary 9 *Outerplanar graphs have integral Fary embeddings.*

4 Integral Convex Embeddings in $\mathbb{R}^3$

In this section, we prove a result weaker than Conjecture 1. A *convex embedding* is an embedding of a planar graph in $\mathbb{R}^3$ such that the set of edges can be extended to form the skeleton of a convex polyhedron on the same set of vertices. One notable property of such an embedding is that it is also *linkless* (and furthermore flat). That is, the set of cycles are pairwise unlinked in a convex embedding. We prove that all planar graphs have integral convex embeddings by using a known sufficient condition for independence in $\mathbb{R}^3$.

Theorem 10 (Gluck [5]) *Let (G, p) be a framework in $\mathbb{R}^3$ such that G is planar and p is a regular point. Then every edge is non-redundant.*

Since a convex embedding stays a convex embedding under small perturbations, we can make the following analogous statements to Theorems 2 and 4.

Theorem 11 *Let (G, p) be a framework in $\mathbb{R}^3$ such that p is a convex embedding and a regular point, and let uv be a non-redundant edge. Then there exists a framework (G, p') such that p' is a convex embedding and a regular point, $||p'(u) - p'(v)||$ is rational, and all other edge lengths remain fixed.*

Theorem 12 *All planar graphs have an integral convex embedding.*

Ziegler [11, Problem 4.18] asks whether every 3-polytope has a realization where every edge has rational length. Theorem 12 answers this in the affirmative in the case where the number of edges is maximal. In particular, the technique of perturbing each edge does not necessarily preserve the flatness of a non-triangular face, so we can only answer this problem for 3-polytopes with only triangular faces.

5 Acknowledgements

I would like to thank Dylan Thurston for a helpful discussion on the topic and the anonymous reviewer for his or her suggestions on the paper. I especially want to thank David Surowski, who in his final piece of academic advice, urged me to submit this paper.

References

[1] L. Asimow and B. Roth, *The rigidity of graphs*, Trans. Amer. Math. Soc. 245 (1978), 279-89.

[2] T. Biedl, *Drawing some planar graphs with integer edge-lengths*, to appear in Canadian Conference on Computational Geometry (2011).

[3] I. Fáry, *On straight-line representation of planar graphs*, Acta Sci. Math. 11 (1948), 229-233.

[4] J. Geelen, A. Guo, D. McKinnon, *Straight line embeddings of cubic planar graphs with integer edge lengths*, J. Graph Theory 58 (2008) No. 3, 270-274.

[5] H. Gluck, *Almost all simply connected closed surfaces are rigid*, Geom. Top., Lecture Notes in Math. 438 (1975), 225-239.

[6] J. Graver, B. Servatius, H. Servatius, *Combinatorial rigidity*, Grad. Stud. Math., Vol. 2, Amer. Math. Soc. (1993).

[7] H. Harborth, A. Kemnitz, M. Moller, A. Sussenbach, *Ganzzahlige planare Darstellungen der platonischen Korper*, Elem. Math. 42 (1987), 118-122.

[8] A. Kemnitz, H. Harborth, *Plane Integral Drawings of Planar Graphs*, Discrete Math. 236 (2001), 191-195.

[9] G. Laman, *On graphs and rigidity of plane skeletal structures*, J. Eng. Math. 4 (1970), 331-340.

[10] K. Wagner, *Über eine Eigenschaft der ebenen Komplexe*, Math. Ann. 144 (1937), 570-590.

[11] G. Ziegler, *Lectures on polytopes*, Springer (1995), 123.