A Randomly Embedded Random Graph is Not a Spanner

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Abstract

Select n points uniformly at random from a unit square, and then form a random graph on these points by adding an edge joining each pair independently with probability p. We show that for every fixed $\epsilon > 0$, if $p < 1 - \epsilon$, then with probability approaching 1 as n becomes large, the resulting embedded graph has unbounded stretch factor.

1 Introduction

Select n points uniformly at random from a unit square, and then form a random graph G on these points by joining each pair independently with probability $p = p(n)$. This is not a "random geometric graph" in the usual sense of that term, because points are connected without regard to their geometric distance. For every two points u and v, let $d(u,v)$ denote their Euclidean distance. Make G weighted by putting weight $d(u,v)$ on every edge uv. For two vertices u and v, let $d_G(u,v)$ denote their shortest-path distance on (weighted) G, and let $d_G(u,v) = \infty$ if there is no (u,v)-path in G. The *stretch factor* of G is defined as

$$\max \frac{d_G(u,v)}{d(u,v)},$$

where the maximum is taken over all vertices u, v. In the open problem session of CCCG 2009 [1], O'Rourke asked if for $p > \ln n/n$, the resulting graph has a bounded stretch factor. We give a negative answer to this question. More precisely, we show that for every fixed $\epsilon > 0$, if $p < 1-\epsilon$, then with probability approaching 1 as n becomes large, the resulting graph has unbounded stretch factor.

2 Proof of the Main Result

Assume that n points are chosen independently and uniformly from a unit square, where n is sufficiently large. Let $p < 1 - \epsilon$ for some fixed $\epsilon > 0$, and build the graph G as in the introduction. In the following, *with high probability* means with probability $1 - o(1)$, where the asymptotics is with respect to n. Fix a positive λ. We will show that with high probability the graph G has stretch factor larger than λ.

Let m be a positive integer satisfying $m^2 \leq n/2 < 2m^2$. Partition the unit square into m^2 squares of side $\frac{1}{m}$ by drawing $m-1$ equally spaced vertical lines and $m-1$ equally spaced horizontal lines. We will call the generated squares of side $\frac{1}{m}$ the *small* squares. Let K be the number of small squares that contain exactly two points. The probability that some point lies on the boundary of some small square is zero, and we will assume that this does not happen.

Lemma 1 *With high probability $K \geq e^{-8}n$.*

Proof. Number the small squares arbitrarily from 1 to m^2. We have

$$K = K_1 + K_2 + \cdots + K_{m^2},$$

where K_i is the indicator variable for the event that the i-th small square contains exactly two points. Hence $\mathbb{E}K_i$ is the probability of this event. Let $1 \leq i \leq m^2$ be arbitrary. The probability that a random point lies in the i-th small square is $1/m^2$. So the probability that exactly two of the n random points are in this square is

$$\mathbb{E}K_i = \left(\frac{n(n-1)}{2}\right)\left(\frac{1}{m^2}\right)^2\left(1-\frac{1}{m^2}\right)^{n-2}.$$

By the choice of m, we have $m^4 \leq n^2/4$ and $m^2 \geq n/4$. These bounds together with the fact that for large n, $\exp(-5/n) \leq 1 - 4/n$ give

$$\mathbb{E}K_i \geq \left(\frac{n^2}{4}\right)\left(\frac{4}{n^2}\right)\left(1-\frac{4}{n}\right)^n \geq e^{-5}.$$

Thus by linearity of expectation,

$$\mathbb{E}K = \mathbb{E}K_1 + \mathbb{E}K_2 + \cdots + \mathbb{E}K_{m^2} \geq m^2 e^{-5} \geq ne^{-7}.$$

Now, we estimate $\mathrm{Var}(K)$ and show that $\mathrm{Var}(K) = O(n)$. Let i, j be arbitrary, with $1 \leq i < j \leq m^2$. The probability that both the i-th square and the j-th square contain exactly two points is

$$\mathbb{E}[K_iK_j] = \binom{n}{2}\binom{n-2}{2}\left(\frac{1}{m^2}\right)^4\left(1-\frac{2}{m^2}\right)^{n-4},$$

and we have

$$\mathbb{E}K_i\mathbb{E}K_j = (\mathbb{E}K_i)^2 = \binom{n}{2}^2\left(\frac{1}{m^2}\right)^4\left(1-\frac{1}{m^2}\right)^{2(n-2)}.$$

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Thus,

$$\begin{aligned}\mathrm{Cov}(K_i, K_j) &= \mathbb{E}[K_iK_j] - \mathbb{E}K_i\mathbb{E}K_j \\ &\le \binom{n}{2}^2 \left(\frac{1}{m^2}\right)^4 \left[\left(1-\frac{2}{m^2}\right)^{n-4} - \left(1-\frac{1}{m^2}\right)^{2(n-2)}\right].\end{aligned}$$

Moreover, since $m^2 = \Theta(n)$,

$$\begin{aligned}&\left(1-\frac{2}{m^2}\right)^{n-4} - \left(1-\frac{1}{m^2}\right)^{2(n-2)} \\ &= \exp\left(\frac{-2(n-4)}{m^2}\right) - \exp\left(\frac{-2n+4}{m^2}\right) + O(1/m^2) \\ &= \exp\left(-\frac{2n}{m^2}\right)\left(e^{8/m^2} - e^{4/m^2}\right) + O(1/m^2) \\ &= e^{\Theta(1)}O(1/m^2) + O(1/m^2) = O(1/m^2).\end{aligned}$$

Thus,

$$\begin{aligned}&\mathrm{Cov}(K_i, K_j) \\ &\le \binom{n}{2}^2 \left(\frac{1}{m^2}\right)^4 \left[\left(1-\frac{2}{m^2}\right)^{n-4} - \left(1-\frac{1}{m^2}\right)^{2(n-2)}\right] \\ &= O(n^4)O(1/m^8)O(1/m^2) = O(1/m^2).\end{aligned}$$

Consequently, since $\mathrm{Var}(K_i) = \mathbb{E}K_i(1-\mathbb{E}K_i) \le 1/4$,

$$\begin{aligned}\mathrm{Var}(K) &= \sum_i \mathrm{Var}(K_i) + \sum_{i\ne j}\mathrm{Cov}(K_i, K_j) \\ &\le m^2/4 + 2\binom{m^2}{2}O(1/m^2) = O(m^2) = O(n).\end{aligned}$$

Let $t = \left(e^{-7} - e^{-8}\right)n$. Then Chebyshev's inequality gives

$$\begin{aligned}\mathbf{Pr}\left[K < ne^{-8}\right] &\le \mathbf{Pr}\left[|K - \mathbb{E}K| \ge t\right] \\ &\le \frac{\mathrm{Var}(K)}{t^2} = o(1).\end{aligned}$$

Thus, with high probability, $K \ge e^{-8}n$. □

A small square S with exactly two points u and v is called *nice* if the following statements are true.

1. Points u and v lie in the circle which is co-centric with S and has radius $r = (7m\lambda)^{-1}$.

2. The points u and v are nonadjacent in graph G.

We claim that the existence of a nice square S implies that the stretch factor of G is larger than λ. In fact, the (weighted) distance between u and v in G is at least $\frac{1}{2m} - r$, since any (u, v)-path in G should go out of S at the very first step. However, the Euclidean distance between u and v is at most $2r$, and we have

$$\left(\frac{1}{2m} - r\right) > \lambda(2r).$$

Let A be the (random) set of small squares that contain exactly 2 points. Let $S \in A$ with points u and v inside it. Then for S to be nice, u and v should lie in the co-centric circle with radius r, and u and v should be nonadjacent in G. The probability of the former is $(\pi r^2 m^2)^2 = (\pi/7\lambda)^2$, and the probability of the latter is $1-p$. These two events are independent, so the probability that S is not nice is $1-(\pi/7\lambda)^2(1-p)$.

Let A_0 be a fixed set of small squares, and assume that we condition on A being equal to A_0. Then the events happening inside each square of A_0 are independent of the others. In particular, the events

$$\{S \text{ is nice} : S \in A_0\}$$

are mutually independent, hence the probability that no nice square exists is equal to

$$\left[1-(\pi/7\lambda)^2(1-p)\right]^{|A_0|}.$$

Therefore, conditioned on the event $|A| \ge e^{-8}n$, the probability that no nice square exists is at most

$$\left[1-(\pi/7\lambda)^2(1-p)\right]^{e^{-8}n},$$

which, since $p < 1-\epsilon$, approaches 0 as n becomes large. By Lemma 1, with high probability the size of A is at least $e^{-8}n$, i.e. the event $|A| \ge e^{-8}n$ happens with probability $1-o(1)$. Thus with high probability a nice square exists and the stretch factor is larger than λ.

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References

[1] Erik D. Demaine and Joseph O'Rourke. Open Problems from CCCG 2009. In *Proceedings of the 22nd Canadian Conference on Computational Geoemtry (CCCG 2010)*, Winnipeg, Manitoba, Canada, August 9–11, 2010, 83–86.