Touring Convex Bodies – A Conic Programming Solution

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Abstract

We study the problem of finding a shortest tour visiting a given sequence of convex bodies in $\mathbb{R}^d$. To our knowledge, this is the first attempt to attack the problem in its full generality: we investigate high-dimensional cases ($d \geq 2$); we consider convex bodies bounded by (hyper)planes and/or (hyper)spheres; we do not restrict the start and the goal positions of the tour to be single points, we measure the length of the tour according to either Euclidean or L_1 metric. Formulating the problem as a second order cone program (SOCP) makes it possible to incorporate distance constraints, which cannot be handled by a purely geometric algorithm.

We implemented the SOCP in MATLAB and obtained its solution with the SeDuMi package. We ran computational experiments, which suggest that the proposed solution is practical.

Finally, we present NP-hardness results, showing that the assumptions we make in the statement of our problems are crucial for the problems to be tractable.

1 Introduction

We consider the problem of finding a shortest tour of a sequence of bodies in $\mathbb{R}^d$. The difference between the problems studied here and classical TSP-like problems is that we assume that the sequence in which the bodies are to be visited is given in advance. An example of such touring problem is the problem of finding a shortest tour through a set of line segments [5, 9, 11, 12]. Due to the bit complexity of the solution, one can only hope to solve the touring problem approximately in polynomial time. A singly-exponential time *exact* algorithm for the problem of touring line segments in $\mathbb{R}^3$ is given in [11]. In [2] the problem of finding ϵ-approximate shortest tour of *lines* in $\mathbb{R}^3$ is solved in time doubly logarithmic in $1/\epsilon$. The general problem for *convex polygons* in $\mathbb{R}^2$ is solved in [4].

SOCP is known to be applicable to a number of computational geometry problems, such as finding extremal volume ellipsoids, centering, separation and classification, placement and facility location, projection and distance problems, intersection and containment of polyhedra, floor planning [1], architectural design [10]; see also [15]. SOCP also provides a natural framework to attack geometric problems in which the goal is to optimize the length of a network (embedding of a planar graph), possibly, under linear and quadratic constraints. A classical example is the Weber (Facility Location) problem [3, 7, 14]: the total length of a *star* is minimized, when the locations of degree-1 nodes of the star are given. In this work we use SOCP to minimize the total length of a *path*, when the nodes of the path are constrained to stay within convex regions. SOCP formulation also allows one to incorporate certain length and distance constraints. A framework similar to ours is outlined in [1, page 433], where it is applied to placement and location problems.

2 SOCP Formulation

We assume that each body $\mathcal{B}_i \,(i = 1 \dots K)$ in the sequence is given as the intersection of a set of J_i bounding (hyper)halfspaces and (hyper)spheres: $\mathcal{B}_i = \{x \in \mathbb{R}^d \,|\, x \in \mathcal{H}_{ij}^-, j = 1 \dots J_i\}$. Each of the bounding surfaces gives rise to a linear or conic constraint $x_i \in \mathcal{H}_{ij}^-$, where x_i is the i-th vertex of the path. Then the touring problem may be formulated as the following SOCP:

$$\begin{array}{rll} \text{minimize} & t_1 + t_2 + \dots + t_{K-1} & \\ \text{subject to:} & t_i \geq \|x_{i+1} - x_i\| & i = 1 \dots K-1 \\ x_i \in \mathcal{H}_{ij}^- & i = 1 \dots K & j = 1 \dots J_i \end{array}$$

If the bodies in the sequence have in total n constraints, then the SOCP allows us to find an ϵ-approximate tour in $O\left(d^3 n^{1.5} K^2 \log \frac{1}{\epsilon}\right)$ ([6]).

2.1 Additional Constraints

In some applications it is natural to ask that the length of each link of the tour does not exceed a certain bound L_i. Sometimes, also a set $C = \{c_1, \dots, c_M\}$ of M control points is given, with the requirement that (some of) the bends of the tour occur close to (some of) the control points: $\|x_i - c_m\| \leq d_{im}$, where $d_{im} \,(i = 2 \dots K-1,\, m = 1 \dots M)$ are some constants.

Imposing any of the above constraints makes it unlikely that the problem can be efficiently solved by

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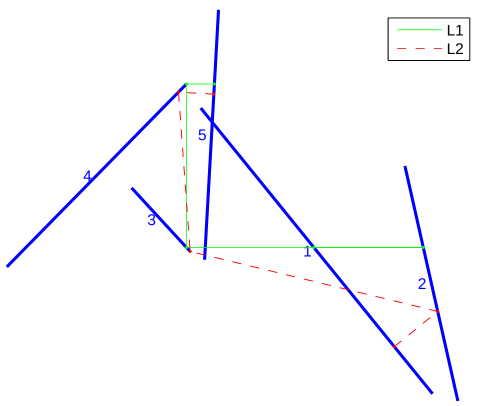

Figure 1: Tours, optimal in different metrics.

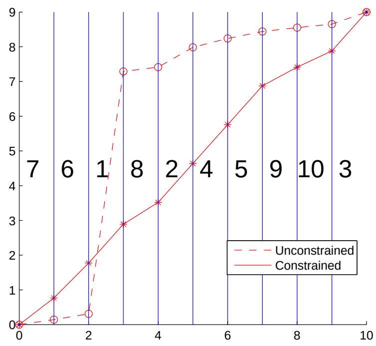

Figure 2: Illustration of two optimal paths through weighted strips, showing the constrained (solid) and unconstrained (dashed) optimal routes. Here, the length constraint is $L = 1.5$. The bold number in each strip is its weight.

purely geometric techniques, like the ones in [9] and subsequent papers on the WRP (see [8]). At the same time, these additional constraints are conic and thus naturally can be handled by our program.

2.2 L_1 metric

Our SOCP can also be applied to the touring problem when the length of the tour is measured according to the L_1 metric. It requires only a slight change in the SOCP: for each link of the tour one variable per dimension is introduced. In $\mathbb{R}^2$, e.g., the new SOCP will be:

$$\begin{array}{lll} \text{minimize} & t_1^x + t_1^y + \ldots + t_{K-1}^x + t_{K-1}^y & \\ \text{subject to:} & (x_i, y_i) \in \mathcal{H}_{ij}^- & j = 1 \ldots J_i \\ & t_i^x \geq ||x_{i+1} - x_i|| & i = 1 \ldots K-1 \\ & t_i^y \geq ||y_{i+1} - y_i|| & i = 1 \ldots K-1 \end{array}$$

Figure 1 shows the tours of a sequence of line segments, optimal under L_1 and L_2 metrics.

2.3 Weighted Links

It is straightforward to modify our solution so that it handles the weighted version, in which each link is assigned a weight. If $w_1, \ldots, w_{K-1}$ are the weights of the links, the objective function changes to $\min w_1 t_1 + \ldots + w_{K-1} t_{K-1}$; the rest of the SOCP remains the same.

Figure 2 shows the optimal weighted tours of a sequence of parallel line segments. Without the constraint on the length of the links, the path obeys Snell's Law of Refraction; the behavior of the constrained path is different.

3 Computational Experiments

We implemented the described program in MATLAB. The solution to the SOCP was obtained with the SeDuMi package by Jos Sturm [13]. We report here the run times for the simplest case, when the bodies are parallel straight line segments of equal length – edges in a weighted subdivision of a box (see Fig. 2).

Theoretically, the running time of the algorithm is $O(K^{3.5} \log 1/\epsilon)$ to achieve accuracy ϵ. We did not change the default SeDuMi setting $\epsilon = 10^{-9}$ in our experiments. We were able to solve instances of the problem with K up to 5000. The SOCP algorithm performed about 15–25 iterations in every instance of the problem. This coincides with the observation, made by Lobo et al. in [6] about primal-dual interior point method for SOCP: the typical number of iterations ranges between 5 and 50, almost independent of the problem size.

The average (over about 100 runs) actual running time of the algorithm for different problem sizes is presented in Figure 3.

Figures 4, 5, 6 and 7 show solutions of the general touring problems in 2 and 3 dimensions.

4 Hardness Results

We complement our solution with NP-hardness results, showing that the assumptions we make in the statement of the problem are crucial to the efficient solvability.

First of all, if the order in which the bodies are to be visited is not given, then our touring problem becomes TSP with neighborhoods (see [8, Ch. 7.4]) and thus is NP-hard.

If the bodies in the sequence are not convex, then our problem is NP-hard by the reduction presented in [4].

Finally, if the length of each link of the tour is bounded from *below*, then even the simplest version of our problem is weakly NP-hard. Indeed, consider the problem of finding a shortest path visiting a sequence

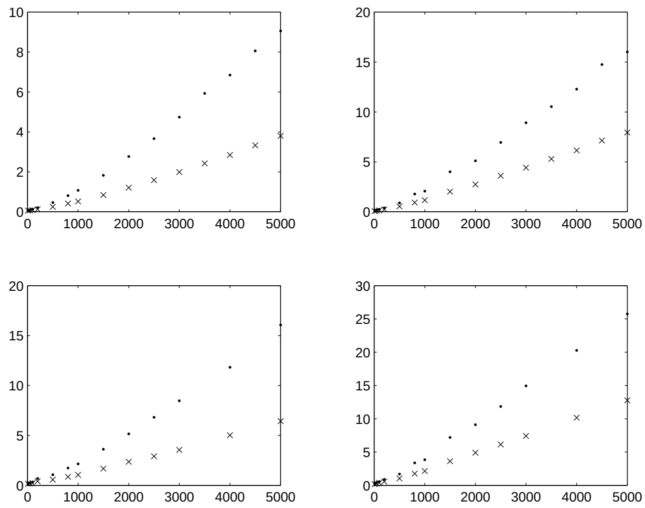

Figure 3: Running times, Windows Machines. Top: 1.9MHz, 512M RAM Compaq laptop; bottom: 1.7MHz, 256M RAM Dell desktop. Left: unconstrained; right: constrained. Dots – run time, sec; crosses – run time per iteration, .1 sec.

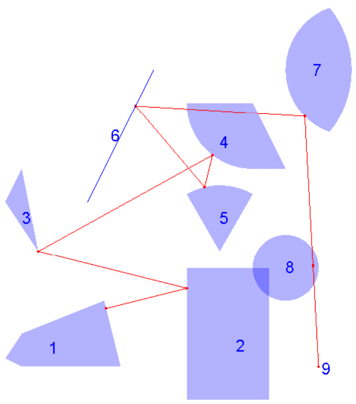

Figure 4: A path in 2D.

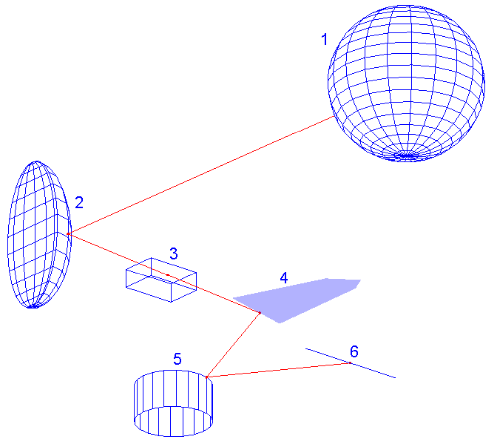

Figure 5: A path in 3D.

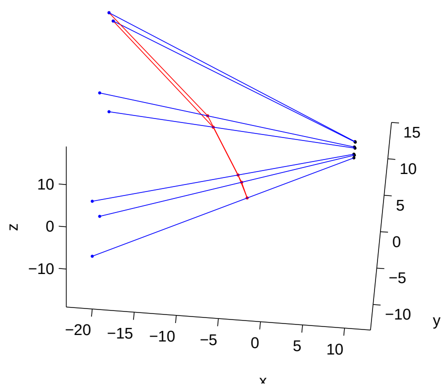

Figure 6: Touring lines in 3D; path endpoints are given.

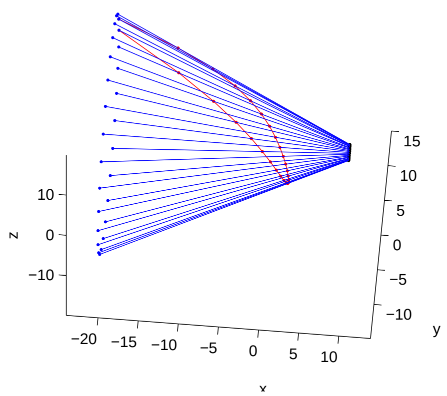

Figure 7: Touring lines in 3D; path endpoints are given.

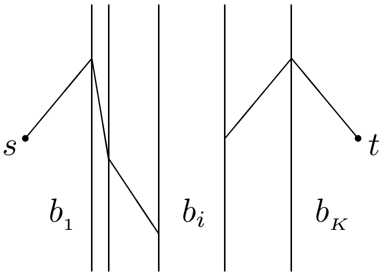

Figure 8: The hardness reduction. The width of strip i is $b_i = \sqrt{L - a_i^2}$. If the ith link has positive slope, then a_i is chosen to be in the subset A'.

of parallel line segments of equal length, with a given start point s and goal point t.

We use a reduction from PARTITION: Given a set $A = \{a_1, \dots, a_K\}$ of K integers summing to S, is there a subset $A' \subset A$ of elements summing to $S/2$? Given an instance of PARTITION, we construct an instance of the touring problem such that the PARTITION problem has answer "Yes" if and only if the optimal path has length at most $(K+1)L$, where L the lower bound on the length of the links (common for all links). See Figure 8.

Theorem 1 *The optimal touring problem is weakly NP-hard if a lower bound is specified for the length of each segment of the path.*

5 Discussion and Open Problems

Our results on the touring problem may be applicable to other optimal path problems and, in particular, may be useful in computing locally optimal paths, as a subproblem in solving TSP with neighborhood problems in which the order is not given.

Since we use SOCP as a "black box", any future improvement in SOCP algorithms will mean corresponding improvement of our solution. In particular, there is a readily available "warm start" initial solution to our SOCP: a random point inside each of the bodies provides a feasible solution to the program.

The difference in the complexity of the touring problem with *upper* and *lower* bounds on the length of the links may be explained as follows. The *upper bound* on a link length induces a convex constraint, the *lower bound* induces a non-convex constraint. Indeed, the set $\mathcal{C}^{\leq} = \{(x, y) \in \mathbb{R}^{2d} \mid \|x - y\| \leq 1\}$ is convex since for $(a, b), (c, d) \in \mathcal{C}^{\leq}$, $\|\frac{a+c}{2} - \frac{b+d}{2}\| \leq \frac{\|a-b\|}{2} + \frac{\|c-d\|}{2} \leq 1 \Rightarrow (\frac{a+c}{2}, \frac{b+d}{2}) \in \mathcal{C}^{\leq}$. At the same time the set $\mathcal{C}^{\geq} = \{(x, y) \in \mathbb{R}^{2d} \mid \|x - y\| \geq 1\}$ is non-convex since already its intersection with the hyperplane $x = 0$ (the exterior of the unit sphere in $\mathbb{R}^d$) is not convex.

Obviously, in some instances the optimal tour is self-intersecting. What is the hardness of finding an optimal *simple* tour (or cycle)? What if each body is just a single point in 2D?

Another open problem ([4]) is the hardness of touring a sequence of *disjoint* non-convex bodies in $\mathbb{R}^2$.

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