Collection depots location problem in the plane

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Abstract

In this paper we consider an extension of the classical facility location problem where besides n customers, a set of p collection depots are also given. In this setting the service of a customer consists of the travel of a server to the customer and return back to the center via a collection depot. We have analyzed the problem and showed that the collection depots problem can be transformed to $O(n^2p^2)$ number of different classical facility location problems and this bound is tight.

1 Introduction

Given is a set of customers or demand points $C = \{c_1, c_2, \ldots, c_n\}$ where each customer c_i is associated with weight w_i. Also given is a set of collection depots $D = \{d_1, d_2, \ldots, d_p\}$. A facility serving a customer dispatches a vehicle that visits the customer and returns to the facility through the collection depot which provides the shortest route. The goal is to minimize the travelled distance. The objective function to be minimized depends on the application. One of the widely used objective functions is to locate the facility at a point that minimizes the maximum of the weighted distance of the round-trip to all the customers. That is, the goal is to minimize $F(s)$, where

$$F(s) = \max_{i=1}^{n} w_i.\{d(s, c_i) + \min_{1\le j\le p}\{d(c_i, d_j) + d(d_j, s)\}\}.$$

Here $d(a, b)$ indicates the Euclidean distance between the points a and b. This problem is known as the 1-center or MinMax collection depot problem.
Another objective function is $G(s)$, where

$$G(s) = \sum_{i=1}^{n} w_i.\{d(s, c_i) + \min_{1\le j\le p}\{d(c_i, d_j) + d(d_j, s)\}\}.$$

This problem is known as the 1-median or the MinSum collection depot problem.

We see that the depot associated with a customer varies as the service center is moved. Let I_s denote the assignment vector of length n where $I_s[i]$ indicates the depot assigned to customer c_i when the service facility is at s. In this case $F(s)$ and $G(s)$ can be rewritten as

$$F(s) = \max_{i=1}^{n} w_i.\{d(s, c_i) + \{d(c_i, I_s[i]) + d(I_s[i], s)\}\} \ldots (1)$$

and

$$G(s) = \sum_{i=1}^{n} w_i.\{d(s, c_i) + \{d(c_i, I_s[i]) + d(I_s[i], s)\}\} \ldots (2)$$

Note that the assignment vector is the same for a particular s for both the objective functions $F(s)$ and $G(s)$.

The collection depot problem was first introduced by Drezner and Wesolowsky [1]. The MinMax and MinSum collection depot problems are essentially generalized versions of the classical MinMax and MinSum facility location problems respectively (consider the case when every client also coincides with a collection depot). Different variations of this problem can be defined, depending on the distance metric used. Several applications are described in Drezner and Wesolowsky [1, 2], such as a septic tank cleaning service, garbage collection or tree pruning service. In each of these problems either the MinMax or the MinSum objective is appropriate. One may wish to minimize the total operation cost, in which case the MinSum objective is appropriate, or one may wish to minimize the largest service time, in which case the MinMax objective is the right one to apply.

It is natural to ask how many different values for the depot assignment vector I_s exist for any placement of the facility in the plane. An obvious upper bound is $O(p^n)$, but tighter bounds should exist. Drezner and Wesolowsky [1] left this question open. In this paper we show that the bound is $O(n^2p^2)$ and it is tight in the worst case. In addition, the depots assignment vectors can be generated in $O(p^2n^2 \log(pn))$ time. Thus the collection depots problem can be transformed to $O(p^2n^2)$ classical MinMax and MinSum facility location problems. Tamir and Halman [8] gave an $O(p^2n^2 \log^3(pn))$ algorithm for the MinMax collection depots problem using the parametric search technique [6]. In this note a

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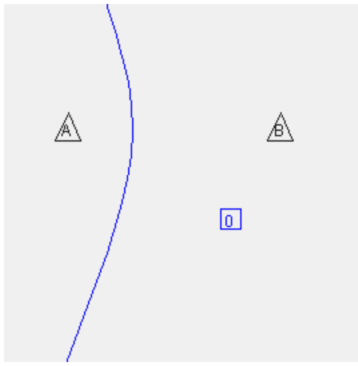

Figure 1: The bisector is an arm of the hyperbola; depots are drawn as triangles, and clients as rectangles

better understanding of the MinMax collection depots problem is provided. Classical MinSum has been shown to be not exactly solvable. However, many practical numerical methods exist. For the first time, the MinSum collection depots problem can now be solved using the classical MinSum algorithm as a subroutine.

2 General Properties

In this section some elegant properties of the collection depots problems are described. A customer c and depots d_1 and d_2 partition the plane into two regions R_1 and R_2 in such a way that for any point q in R_1, the round trip from q to c through d_1 is smaller than the round trip from q to c through d_2, and similarly for any point q in R_2, the round trip from q to c through d_2 is smaller than the round trip from q to c through d_1. Let the curve that partitions the plane into R_1 and R_2 be denoted by η which may not be a straight line. Note that any point q on η, that is on the boundary of R_1 and R_2, must satisfy the following equation

$$d(q,c)+d(c,d_1)+d(d_1,q) = d(q,c)+d(c,d_2)+d(d_2,q) \ldots (3).$$

We can easily show that

Lemma 1 *The locus of the points satisfying the equation (3) is an arm of a hyperbola (Fig. 1).*

Let us denote the two arms of the hyperbola h as h_1 and h_2 where d_1 and d_2 are two foci of h. Then any point q on h satisfies the equation $(d(q,d_1) - d(q,d_2)) = \alpha$ where $\alpha = d(c,d_2) - d(c,d_1)$ is constant. Note that, if $d(q,d_1) - d(q,d_2)$ is positive for any q on h_1, then $d(q,d_1) - d(q,d_2)$ is negative for any point q on h_2.

Therefore, the following observations can be made.

Observation 1 *η does not contain both the arms h_1 and h_2 of the hyperbola h.*

Observation 2 *If η contains arm h_1 of the hyperbola h then customer c must be on arm h_2.*

Without any loss of generality, assume that $d(c,d_2) - d(c,d_1) \geq 0$. Using the triangular inequality, we can say that $d(c,d_2) - d(c,d_1) \leq d(d_1,d_2)$. Note that when $d(c,d_2) - d(c,d_1) = d(d_1,d_2)$ then c, d_1 and d_2 are collinear and d_1 is in between c and d_2. In that case, for any q,

$$d(q,c) + d(c,d_1) + d(d_1,q) \leq d(q,c) + d(c,d_2) + d(d_2,q)$$

which implies that the the region R_2 has an empty interior. The face separating R_1 and R_2 is basically the half line $[d_2,\infty)$ on the straight line l defined by c, d_1 and d_2.

Hence

Observation 3 *When $d(c,d_2) - d(c,d_1) = d(d_1,d_2)$ then η is a half line $[d_2,\infty)$ on the straight line l.*

¿From the above observations, the following lemma can be established.

Lemma 2 *η is either a half line $[d_2,\infty)$ on the straight line l or an arm of the hyperbola h.*

Lemma 3 *If two hyperbolas have the same set of foci, then either they coincide or they are parallel,* i.e. *they never intersect properly.*

Let d_1 and d_2 be two depots. Let η_i denote the separating hyperbolic arm separating regions R_1^i and R_2^i defined for customer c_i (as before, a facility in R_1^i uses depot d_1 for client c_i rather than d_2). From the above lemma we can conclude that

Lemma 4 *The hyperbolic arms $\eta_i, i = 1,2,\ldots,n$ do not properly intersect, and therefore they partition the plane into $n+1$ regions such that a service center in any of these regions fixes the optimal choice of the depot relative to d_1 and d_2 for each customer.*

Let us denote H_{ij} as the set of n non intersecting hyperbolic arms determined by depots d_i and d_j and the set of customers C. As any two elements of the set H_{ij} cannot intersect and two elements of distinct sets can intersect at most two times, we can conclude the following lemma.

Lemma 5 *Two sets of mutually non intersecting hyperbolic arms H_{ij} and H_{kl} intersect in at most $O(n^2)$ points.*

Since $p(p-1)/2$ possible unordered pairs of depots generate sets $\{H_{ij} | i = 1,2,\ldots,p, j = 1,\ldots,i-1\}$ where each one is a collection of $O(n)$ non intersecting hyperbolic arms, we can conclude the following theorem.

Theorem 6 *At most $O(n^2p^4)$ different feasible assignments of depots are possible for any choice of a service center in the Euclidean metric.*

2.1 An Improvement

In this section we show that the bound in Theorem 1 can be improved to $O(n^2p^2)$ and this bound is tight in the worst case.

Let c be a customer of C. For each d_i of D, we are interested in computing $V_c(D, d_i)$ which is the locus of points in the plane such that for any service center in $V_c(D, d_i)$, the shortest tour to c uses the depot d_i among all depots in D, i.e.

$$V_c(D, d_i) = \{x | d(x, c) + d(c, d_i) + d(d_i, x) \leq d(x, c) + d(c, d_j) + d(d_j, x) \quad \forall j\}$$

Lemma 7 *Each $V_c(D, d_i)$ is unbounded.*

Proof: Consider the Voronoi polygon $V_c(D, d_i)$. Consider the ray starting from d_i in the direction of $\vec{cd_i}$. Clearly for any facility z on the ray, the nearest depot for the customer c is d_i. Therefore, $z \in V_c(D, d_i)$. $\square$

As a consequence of the above lemma we can claim that

Theorem 8 *$V_c(D) = \cup_{i=1}^{p} V_c(D, d_i)$ for each $c \in C$ partitions the plane into exactly p regions and contains at most $2p-4$ vertices and $3p-6$ edges (hyperbolic arcs).*

These diagrams $V_{c_i}(D), i = 1, 2, \ldots n$ need to be merged together to determine the set of all feasible assignments of depots. Since each edge in $V_{c_i}(D)$ can intersect all the edges in $V_{c_j}(D), \forall c_j \neq c_i$ in the worst case, Theorem 6 can be improved as follows

Theorem 9 *At most $O(n^2p^2)$ different feasible assignment of depots is possible for any choice of a service center in Euclidean metric.*

It is possible to construct an example to show that the above bound is tight. First, the arrangement consists of a vertical line of depots. A single customer c to the right of the lowest depot produces $V_c(D)$ of Figure 2 Depots are displayed as triangles, and customers are displayed as squares.

Adding a second customer to the right of the highest depot produces Figure 3. The idea can be extended by adding additional customers to the right of the existing customers (Figure 4). This way we can generate an example whose number of feasible assignment of depots is $\Theta(n^2p^2)$.

2.2 Computing $V_c(D)$

Let u_i denote the distance of d_i to c. Then we can rewrite $V_c(D, d_i)$ by

$$V_c(D, d_i) = \{x | dist(x, d_i) + u_i \leq dist(x, d_j) + u_j, \ \forall j \in D\}.$$

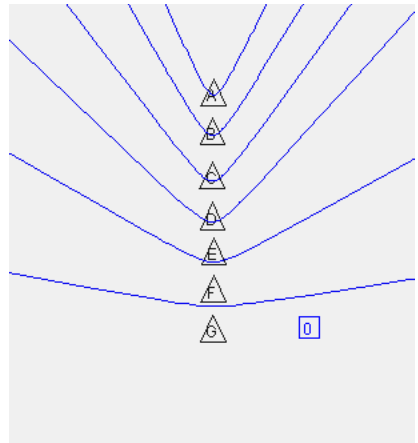

Figure 2: Voronoi diagram involving one customer

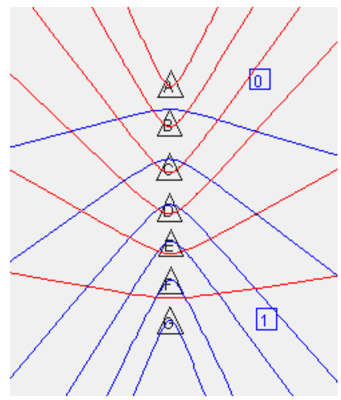

Figure 3: Voronoi diagram involving two customers

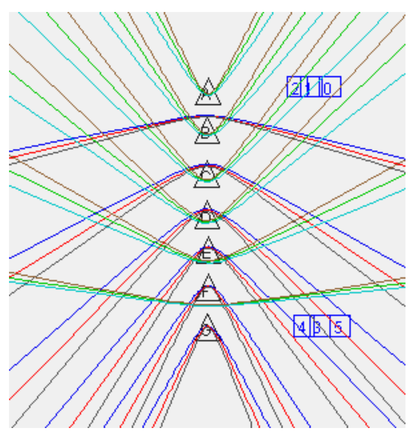

Figure 4: Worst case situation

Thus $V_c(D)$ is the additively weighted Voronoi diagram of D where d_i is associated with the additive weight $dist(c, d_i)$. This diagram can be computed in $O(p \log p)$ time [5]. We can then compute $\cup_{i=1}^{n} V_{c_i}(D)$

in $O(p^2n^2 \log(pn))$ time using the standard plane sweep technique. The resulting merged picture partitions the plane into at most $O(p^2n^2)$ regions such that for any point in a particular region, the depots assignment for the customers of C remains the same. Therefore,

Lemma 10 *All different feasible assignment of depots can be computed in $O(p^2n^2 \log(pn))$ time using $O(pn)$ space.*

3 MinMax problem

¿From the previous theorem we observe that the MinMax collection depots problem can be solved by resolving a slightly extended version of the classical Euclidean weighted MinMax problem for each of the $O(p^2n^2)$ possible feasible assignments of depots. In our case (i.e. when a region is fixed), each customer is associated with a multiplicative as well as an additive weight. A linear time algorithm is known for the classical Euclidean MinMax problem [6] This algorithm can be modified to solve our version of the problem. However, we can do better.

Tamir and Halman [8] presented an $O(p^2n^2 \log^3(pn))$ algorithm for the MinMax collection depots problem. The algorithm is based on the parametric approach of Megiddo [6] which requires an efficient parallel implementation for the following decision problem (called covering problem): Determine whether there exists a facility location such that the maximum round trip cost of the customers of C is at most r.

Let

$$Y_i(r, d_j) = \{x | dist(x, c_i) + dist(x, d_j) \leq \frac{r}{w_i} - dist(c_i, d_j)\}.$$

Here $Y_i(r, d_j)$ represents an ellipse where for any y in the ellipse, the round trip distance from y through c_i and d_j is no more than r. Let $Z_i(r) = \cup_{j=1}^{p} Y_i(r, d_j)$. For the covering problem we ask the question: Is $\cap_{i=1}^{n} Z_i(r)$ empty? It was argued in [8] that the boundary of $Z_i(r)$ can have $O(p2^{\alpha(p)})$ vertices and elliptical arcs where $\alpha(p)$ is the functional inverse of the Ackermann's function. However it can be shown that

Lemma 11 *The size of the boundary of $Z_i(r)$ is $O(p)$.*

Proof: Since in the round-trip from any facility location to a customer c_i uses the depot that minimizes the trip cost, we are only interested in the part of each $Y_i(r, d_j)$ which lies inside $V_{c_i}(D, d_j)$. Let $Y'_i(r, d_j) = Y_i(r, d_j) \cap V_{c_i}(D, d_j), i = 1, 2, \ldots n$, and $Z'_i(r) = \cup_{j=1}^{n} Y'_i(r, d_j), i = 1, 2, \ldots, n$. Clearly, $Y'_i(r, d_j)$ and $Y'_i(r, d_k)$ for any $j \neq k$ can share at most one edge. Therefore, the boundary description of $Z'_i(r)$ is the same as that of $V_{c_i}(D)$, which is $O(p)$. $\square$

As described in [8], whether $\cap_{i=1}^{n} Z'_i(r)$ is non-empty can be tested in $O(p^2n^2 \log(pn))$ (Section 6 in Sharir and Agarwal [7]). The optimal value of the MinMax collection depots problem is the smallest r of the covering problem for which $\cap_{i=1}^{n} Z'_i(r)$ is non-empty. For this we apply the parametric approach of Megiddo [6]. Therefore [8]

Theorem 12 *The optimal solution to the MinMax collection depots problem can be computed in $O(p^2n^2 \log^3(pn))$.*

4 MinSum problem

It was observed in [1] that $G(s)$ in equation (2) can rewritten as follows:

$$\begin{aligned} G(s) &= \textstyle\sum_{i=1}^{n} w_i d(s, c_i) + \sum_{i=1}^{n} w_i (c_i, I_s[i]) \\ &\quad + \textstyle\sum_{i=1}^{n} w_i (I_s[i], s) \\ &= G_1(s) + G_2(s) + G_3(s) \end{aligned}$$

For a given assignment vector I, $G_2(s)$ is constant. Therefore, for a given I, minimizing $G(x)$ is the same as minimizing $G_1(x) + G_3(x)$ which is the classical MinSum problem of $n + p$ variables. Therefore,

Theorem 13 *The MinSum collection depot problem can be solved in $O(p^2n^2)$ times the time it takes to solve the classical MinSum problem.*

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