A vertex-face assignment for plane graphs

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Abstract
For any planar straight line graph (PSLG), there is a vertex-face assignment such that every vertex is assigned to at most two incident faces, and every face is assigned to all its reflex vertices and one more incident vertex. The existence of such an assignment implies, in turn, that any PSLG can be augmented to a connected PSLG such that the degree of every vertex increases by at most two.

1 Introduction

A *planar straight line graph (*PSLG*)* partitions the plane into connected components, which are the *faces* of the graph. Every PSLG has an unbounded *outer face* and, if it has circuits, then it also has *bounded faces*.

Let $V(G)$ and $F(G)$ denote the set of vertices and faces, respectively, of a PSLG G. A *vertex-face assignment* for G is a multiset $A \subset V(G) \times F(G)$, where every pair $(v, f) \in A$ is an incident vertex-face pair. If $(v, f) \in A$, then we say that vertex v *is assigned to* face f, and vice versa, face f *is assigned to* vertex v. Our main result relates vertices and faces through a special type of vertex-face assignment.

Given a PSLG G and a vertex $v_0 \in V(G)$, a vertex-face assignment is a $\star$*-assignment* if it satisfies the following conditions.

(i) Every vertex is assigned to at most two faces[1];

(ii) v_0 is assigned to at most one face;

(iii) Every face is assigned to all its reflex vertices and one additional vertex[2].

Theorem 1 *For every* PSLG G *and* $v_0 \in V(G)$*, there is a* $\star$*-assignment.*

A triangulation on n vertices, for example, has $2n-4$ faces. The outer face has three reflex vertices, all other faces are convex. Therefore, a $\star$-assignment maps the $2n-4$ faces to a total of at least $(2n-4)+3 = 2n-1$ vertices, and the n vertices are assigned to at most $2n-1$ faces. Theorem 1 is similarly tight for *pseudo-triangulations* (PTNs). A PTN is a PSLG where the outer face has no convex vertices, and every bounded face has exactly three convex vertices. A vertex v is *pointed* if it is a reflex vertex for some face. A PTN is *pointed* if *all* vertices are pointed. Consider a PTN G with n vertices, m of which are not pointed. It is easy to see that G has $n+m-1$ faces [12]. The $n+m-1$ faces have $n-m$ reflex vertices in total, and so a $\star$-assignment maps these faces to at least $(n+m-1)+(n-m) = 2n-1$ vertices.

Recently, Hoffmann and Tóth [6] conjectured that every PSLG has a $\star$-assignment with the additional condition that every bounded face is assigned to at least one convex vertex. We show that such an assignment *does not* always exist.

Proposition 2 *There is a* PSLG Γ *with the property that for every* $\star$*-assignment there is a bounded face* $f \in F(\Gamma)$ *which is not assigned to any convex vertex of* f*.*

We can, as well, generate an infinite family of such PSLGs by gluing several copies of Γ together.

1.1 Encompassing graphs

A PSLG is not necessarily connected. An *encompassing graph* for a PSLG G is a *connected* PSLG on the same vertex set that contains all edges of G. Encompassing graphs for PSLGs are similar to spanning trees for planar point sets. The key difference is, though, that the edges connecting disjoint components of G are pairwise non-crossing, and they cannot cross the edges of G, either.

Bose, Houle, and Toussaint [1] proved that any plane straight line matching (i.e., disjoint line segments in the plane) has an encompassing tree of maximum degree at most three. Hoffmann and Tóth [6] have generalized this result and proved that any plane straight line *forest* can be augmented to an encompassing tree such that the degree of every vertex increases by at most two. As a consequence of Theorem 1, we can now extend their result to *arbitrary* PSLG*s*.

Theorem 3 *Any* PSLG G *can be augmented to an encompassing graph of* G *such that the degree of every vertex increases by at most two.*

We say that a graph is *vertex-colored* if every vertex has a color and adjacent vertices have different colors. Hurtado et al. [7] proved that any vertex-colored PSLG with no singletons can be augmented to a vertex-colored encompassing graph. Hoffmann and Tóth [6] proved that any vertex-colored PSLG *forest* (which has exactly one face) with no singletons can be augmented to an encompassing graph while the degree of every vertex increases by at most two. Using Theorem 1, we can extend this result, too, to arbitrary vertex-colored PSLGs.

Theorem 4 *Any vertex-colored* PSLG G *with no singleton component can be augmented to a vertex-colored encompassing graph of* G *such that the degree of every vertex increases by at most two.*

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[1] This includes the possibility of assigning a vertex twice to the same face, since the assignment A may be a multiset.

[2] The additional vertex may be a reflex vertex, in which case the face is assigned twice to one of its reflex vertices.

Related previous work. There seems to be little known about vertex-face assignments. Brooks et al. [2] showed that a $\star$-assignment exists for even degree triangulations. In fact, it is not difficult to construct a $\star$-assignment for *any triangulation*.

2 Proof of Theorem 1

We construct an assignment for a Pslg in four steps. We first prove Theorem 1 for pointed Ptns and combinatorial pointed Ptns. We then extend the assignment for Ptns and for arbitrary Pslgs.

2.1 Pointed pseudo-triangulations

It is well known that *pointed* Ptn*s* are related to rigidity theory. Henneberg [4] defined two (rigidity preserving) operations for *abstract graphs* almost one hundred years ago. Restricting these operations for Pslgs, we can modify a graph only along the boundary of one face. There are two *planar Henneberg operations* for a Pslg:

H1 Consider two vertices u, v along a face f. Add a new vertex w in the interior of f, and split f into two faces by two new edges uw and vw. (Fig. 1)

H2 Consider an edge u_1u_2 and a vertex v along a face f. Replace edge u_1u_2 by a path (u_1, w, u_2) where w is a new vertex, and split face f into two faces by a new edge vw. (Fig. 1, right)

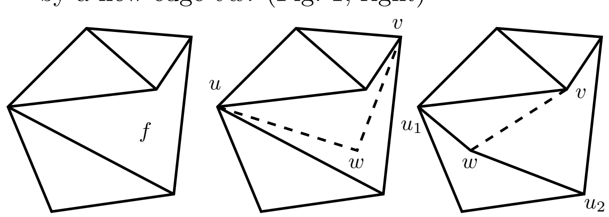

Figure 1: Operations H1 and H2.

Recently, Haas et al. [3] proved a planar decomposition theorem for pointed Ptns.

Theorem 5 (Haas et al. [3]) *Given a pointed* Ptn *G and a vertex $v_0 \in V(G)$, there is a sequence $(G_0, G_1, \ldots, G_k)$ of pointed* Ptn*s such that G_0 is a triangle $u_0v_0w_0$, $G_k = G$, and G_{i+1} is obtained from G_i by a planar Henneberg operation for $i = 0, 1, \ldots, k-1$.*

Lemma 6 *Every pointed* Ptn *has a $\star$-assignment.*

Proof. Consider the sequence $(G_0, G_1, \ldots, G_k)$ of pointed Ptns from Theorem 5. Every vertex in every G_i, $i = 0, 1, \ldots, k-1$, is pointed. For every vertex v, the pair (v, f) where v is reflex for f must be in a $\star$-assignment. By discarding these pairs, we obtain a *reduced $\star$-assignment*, which satisfies the following conditions: (i') Every vertex is assigned to at most one face; (ii') v_0 is not assigned; (iii') every face is assigned to one vertex.

It suffices to find a reduced assignment for every G_i, $i = 0, 1, \ldots, k-1$. The initial graph G_0 has two faces: the outer face f_0 and the triangle f. Let $A'(G_0) = \{(u_0, f_0), (w, f)\}$, which is a reduced $\star$-assignment.

Consider a pointed Ptn G_i with a reduced $\star$-assignment A'_i. We obtain G_{i+1} from G_i through operation H1 or H2. Both H1 and H2 split a face $f \in F(G_i)$ into two faces, f_1 and f_2, and insert a new vertex w incident to both f_1 and f_2. All other faces $f_3 \in F(G_i)$, $f_3 \neq f$, remain incident to all vertices they were incident to in G_i. Suppose A'_i assigns f to a vertex v_1 and w.l.o.g. v_1 is incident to f_1. We obtain a reduced $\star$-assignment $A'_{i+1} = A'_i - \{(v_1, f)\} + \{(v_1, f_1), (w, f_2)\}$ that assigns the new vertex w to f_2. $\square$

2.2 Pointed combinatorial pseudo-triangulations

Haas et al. [3] defined the combinatorial pseudo-triangulation, which replaces the intuitive notion of reflex and convex vertices by abstract ones. An *angle* of a Pslg is a triple (v, e_1, e_2) of a vertex v and two incident edges e_1 and e_2 which are consecutive in the cyclic order of all edges incident to v. Since e_1 and e_2 are on the boundary of a common face, every vertex-face incidence determines (at least one) angle.

A *combinatorial pseudo-triangulation* (Cptn) is a Pslg where every *angle* is labeled either *big* or *small*; every vertex is the apex of at most one *big* angle; every angle of the outer face is labeled *big*, and every bounded face has exactly three angles labeled *small*. In particular, a Ptn with the *natural labeling* (where reflex angles are big and convex angles are small) is a Cptn. A vertex of a Cptn is pointed if it is the apex of a big angle, and a Cptn is pointed if all its vertices are pointed.

Haas et al. [3] proved that every pointed Cptn can be realized (with the same vertex-face incidence structure) as a pointed Ptn. It follows immediately that:

Lemma 7 *Every pointed* Cptn *has a $\star$-assignment*[3]*.*

2.3 Combinatorial pseudo-triangulations

Next, we would like to construct arbitrary Ptns with local planar operations, similarly to the planar Henneberg construction of pointed Ptns. For this, we define one more operation (inserting an edge) and cite a recent result by Orden et al. [10].

H3 Consider two non-adjacent pointed vertices u, v along a face f such that f has a big angle at u. Add a new edge uv that splits f into two faces. Edge uv splits the big angle of u into two small angles; it splits the angle of v into two small angles iff f had a small angle along v[4].

Theorem 8 (Orden et al. [10]) *Given a* Ptn *G and a vertex $v_0 \in V(G)$, there is a sequence $(G_0, G_1, \ldots, G_k)$ of* Cptn*s such that G_0 is a* pointed Cptn *with $v_0 \in V(G_0)$, $G_k = G$, and G_{i+1} is obtained from G_i by operation H1 or H3, for $i = 0, 1, \ldots, k-1$.*

[3]For a Cptn, condition (iii) of $\star$-assignments requires that every face is assigned to the apexes of all its *big* angles and one additional incident vertex.

[4]That is, u becomes non-pointed, and v remains pointed.

(In fact, Orden et al. [10] proved a stronger result: Every *generically rigid* Cptn can be constructed from a pointed Cptn by operations H1 and H3 through a sequence of generically rigid Cptns. And, in particular, every Ptn is a generically rigid Cptn.)

Note that the Cptns are, indeed, necessarily for Theorem 8: The Ptn on the right of Fig. 2 cannot be obtained from another Ptn by operations H1 or H3.

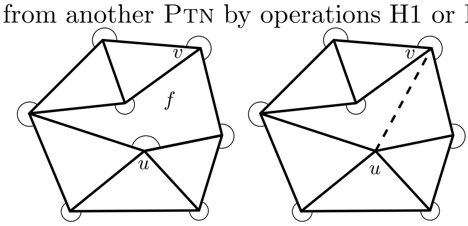

Figure 2: Operation H3. *Big* angles are marked. The Cptn on the right is a Ptn (all big angles are reflex).

Lemma 9 *Every* Cptn *has a $\star$-assignment.*

Proof. Similarly to the proof of Lemma 6, we may ignore the assignment of pointed vertices to the faces at which they are big. It is enough to find a *reduced $\star$-assignment* that satisfies these conditions: (i') Every pointed (non-pointed) vertex is assigned to at most one face (two faces); (ii') If v_0 is pointed (non-pointed) then it is not assigned (assigned to at most one face); (iii') every face is assigned to one vertex.

Consider a Cptn G_i with a reduced $\star$-assignment A'_i. We obtain G_{i+1} from G_i via operation H1 or H3. In case of operation H1, we repeat the argument in the proof of Lemma 6. Operation H3 inserts a new edge uv and splits a face $f \in F(G_i)$ into two faces, f_1 and f_2. All other faces $f_3 \in F(G_i)$, $f_3 \neq f$, remain incident to the same vertices. Suppose A'_i assigns f to a vertex v_1 and w.l.o.g. v_1 is incident to f_1. We obtain a reduced $\star$-assignment $A'_{i+1} = A'_i - \{(v_1, f)\} + \{(v_1, f_1), (u, f_2)\}$ that assigns the new non-pointed vertex u to f_2. □

2.4 Arbitrary Pslgs

Given a Pslg G, we can augment G to a Ptn G' such that every pointed vertex of G remains pointed: One can insert edges recursively so that no new edge crosses any previous edge or partitions any reflex angle into two convex angles [12].

Proof. [of Theorem 1] Consider a Pslg G. We augment G to a Ptn G' by adding a set E of edges while keeping all pointed vertices pointed. The Ptn G' has a $\star$-assignment by Lemma 9. We remove the edges of E one by one and maintain a $\star$-assignment. When we remove an edge uv, let f_1 and f_2 denote the two faces along uv. If $f_1 = f_2$ then we do not change the assignments. If $f_1 \neq f_2$, then the edge removal merges them to a face $f = f_1 \cup f_2$. Every vertex assigned to f_1 or f_2 should now be assigned to f. Since no edge removal merges two convex angles into a reflex angle, the resulting assignment is a $\star$-assignment for G'. □

Computational complexity. For a given *pointed* Cptn with n vertices, Haas et al. [3] can compute a sequence of planar Henneberg operations in $O(n)$ time (in fact, they compute the sequence of reverse operations), and so a $\star$-assignment can also be constructed in $O(n)$ time. For a given Cptn with n vertices, one can compute a sequence of H1 and H3 operations to obtain a pointed Cptn in $O(n)$ time by results of Orden et al. [10] (again, computing the reverse sequence first), and so a $\star$-assignment can also be constructed in $O(n)$ time. Finally for a given Pslg G with n vertices, we can compute a minimal encompassing pseudo-triangulation $T(G)$ in $O(n \log n)$ time (by applying in every face of G a variant of Pocchiola and Vegter's greedy flip algorithm [11, 12]). $T(G)$ has $O(n)$ more edges than G. While deleting these edges one-by-one, we can compute a $\star$-assignment for G from a $\star$-assignment of $T(G)$ in $O(n)$ time. Altogether, the total complexity of our algorithm for computing a $\star$-assignment for a Pslg G on n vertices is $O(n \log n)$.

3 A negative result

In this section, we prove Proposition 2. Consider the pointed Ptn Γ in Fig. 3. We argue that for any $\star$-assignment of Γ, there is a pseudo-triangle face assigned to reflex vertices only.

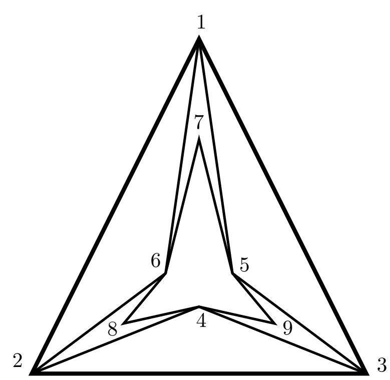

Figure 3: Graph Γ.

Suppose, to the contrary, that every bounded face of Γ is assigned to a convex vertex. Consider the six bounded faces along the triangle 123. All convex vertices of these faces are in the six element set $\{1, 2, 3, 4, 5, 6\}$. But one of $\{1, 2, 3\}$ is assigned twice to the outer face, and it cannot be assigned to any bounded face. A contradiction: A $\star$-assignment cannot assign a convex vertex to one of the quadrilaterals $\{1675, 3594, 2486\}$.

4 Encompassing graphs

Proof. [of Theorem 3.] Consider a Pslg G, and let $C_0, C_1, \ldots, C_{k-1}$ denote its connected components. We augment G to a connected Pslg inductively. During our algorithm we choose a special vertex v_i for every component C_i and a $\star$-assignment A_i for the pairs of C_i and v_i. The following two properties guarantee that the degree of every vertex increases by at most two: (a) for every pair $(v, f) \in A_i$, $i = 0, 1, \ldots, k-1$, we may increase the degree of v by an edge lying in the face f; (b) in addition, we may increase the degree of each special vertex $v_i \in C_i$ by one.

We initiate our inductive algorithm by choosing an vertex $v_0 \in V(G)$ on the convex hull of G, setting B_0 to be the component C_j containing v_o, and computing a $\star$-assignment $A(B_0)$ for B_0 and v_0. One induction step inserts an edge e_i between B_i, $i = 0, 1, \ldots, k-1$, and a component of $G - B_i$. Then it computes a $\star$-assignment for B_{i+1}, which is the component of the augmented graph $G \cup \{e_0, \ldots, e_i\}$ containing vertex v_0. The induction step is performed as follows.

Consider a face f of B_i that contains some component of $G - B_i$. We denote by G_f the graph formed by all components of $G - B_i$ lying in f. Let v_f be a vertex assigned to f by the reduced $\star$-assignment $A'(B_i)$. We search iteratively for a visibility edge $e_i = u_i v_{i+1}$, where $u_i \in V(B_i)$ is either vertex v_f or a reflex vertex of f, and some $v_{i+1} \in V(G_f)$. For this iterative search, we let $g := f$ and $u_i := v_f$.

> `Repeat` until u_i sees some vertex $v_{i+1} \in V(G_f)$ lying in g: Let $\mathrm{Vis}(u_i) \subset g$ be the region visible from u_i where G_f is considered opaque. Let g' be a connected region of $g \setminus \mathrm{Vis}(u_i)$ that contains vertices of both G_f and B_i. The common boundary of $\mathrm{Vis}(u_i)$ and g' contains a reflex vertex u'_i of g. Let $g := g'$ and $u_i := u'_i$.

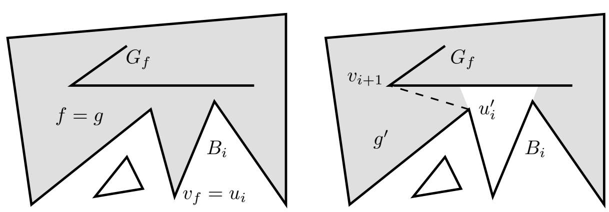

Figure 4: One iteration for g and u_i.

Once we have found edge $e_i = u_i v_{i+1}$ between B_i and a component $C_{i+1} \subset G - B_i$, we compute a $\star$-assignment A_{i+1} for C_{i+1} and v_{i+1}. Let $B_{i+1} = B_i \cup C_{i+1} \cup \{e_i\}$. Every face of B_{i+1} is the face of either B_i or C_{i+1}, with one exception: face f of B_i and the outer face of C_{i+1} correspond to the same face f' in $F(B_{i+1})$. We combine the $\star$-assignments $A(B_i)$ and A_{I+1} into a common $\star$-assignment $A(B_{i+1})$. To face $f' \in F(B_{i+1})$, the reduced $\star$-assignment $A'(B_{i+1})$ maps the vertex of C_{i+1} that A'_{i+1} assigns to the outer face of C_{i+1}.

It is easy to see that the resulting graph $G \cup \{e_i : i = 0, 1, \ldots, k-1\}$ satisfies constraints (a) and (b). This completes the proof of Theorem 3. □

The proof of the colorful variant, Theorem 4, is more involved and is omitted from this abstract. It uses the same technique as [6] (based on multiple visibility sweeps of the faces of the input PSLG G).

5 Open problems

We have constructed a $\star$-assignment for every PSLG. Is there a PSLG with a unique $\star$-assignment? How many $\star$-assignments exist for a given PSLG G and $v_0 \in V(G)$?

The definition of $\star$-assignments poses different conditions for convex and reflex vertices of a PSLG. Is it true that every CPTN has a $\star$-assignment[5]?

For disjoint line segments in the plane, a bounded degree encompassing tree can be used to construct a bounded degree encompassing PTN [5, 9]. Can every PSLG be augmented to a PTN so that the degree of every vertex increases by no more than a constant?

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[5] Remember: for CPTNs, condition (iii) requires that every face is assigned to the apexes of all its *big* angles and one additional incident vertex.