The complexity of domino tiling

Therese Biedl *

Abstract

In this paper, we study the problem of how to tile a layout with dominoes. For non-coloured dominoes, this can be determined easily by testing whether the layout graph has a perfect matching. We study here tiling with coloured dominoes, where colours of adjacent dominoes must match.

It was known that coloured domino tiling is NP-hard when the layout graph is a tree. We first strengthen this NP-hardness result in two ways: (1) we can use a path instead of a tree, or (2) we can force that exactly all given dominoes are used. However, both these reductions (as well as the existing one) use an unbounded numbers of colours, which is not realistic for domino tiling. As our main interest, we hence study domino tiling with a constant number of colours. We show that this is NP-hard even with 3 colours. We prove these results by relating domino tiling to a graph homomorphism problem, which may be of independent interest.

1 Introduction

In this paper, we consider the problem of tiling a given layout with dominoes. A *layout* L is an integral orthogonal polygon, i.e., a polygon for which all edges are horizontal or vertical and vertices are placed with integer coordinates. The layout may have holes. A *domino* is a 1×2-rectangle. A *domino tiling* of L is a placement of dominoes inside L such that no two dominoes intersect in the interior and every point of L is covered by a domino.

A layout L can be described via the *layout graph* G^L, which is defined by adding one vertex for every integral 1×1-square of the layout and connecting adjacent squares by an edge. See also Figure 1. It is folklore that a layout L has a domino tiling if and only if G^L has a *perfect matching*, i.e., a set of edges M such that every vertex is incident to exactly one edge in M, and this can be tested efficiently [4].

In this paper, we study domino tiling while considering colours of dominoes; this was introduced in [5]. Let $C = \{c_1, \ldots, c_l\}$ be a finite set of l colours. A

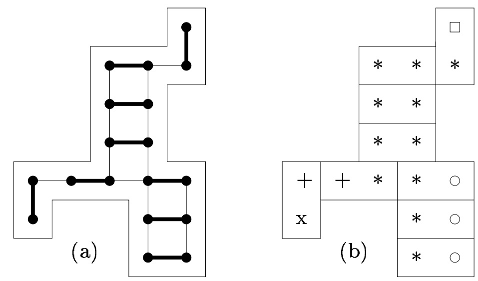

Figure 1: (a) A layout and its layout graph with a perfect matching (bold.) (b) A tiling with coloured dominoes; vertices with the same symbol must be coloured the same.

coloured domino is an unordered pair (c_i, c_j). *Uni-coloured dominoes*, i.e., dominoes for which $c_i = c_j$, are not expressively forbidden, but will not be used in our NP-hardness reductions.

A *coloured domino tiling* of a layout graph G^L is similar to a domino tiling, except that colours of dominoes must match up. Thus, given a multi-set D of coloured dominoes, a *coloured domino tiling* of G^L with D is a perfect matching M in G^L and a colouring $c : V \to C$ such that

- if (v, w) is an edge in M, then $(c(v), c(w))$ is a domino in D,
- if (v, w) is an edge not in M, then $c(v) = c(w)$, and
- every domino is used at most once, i.e., there is an injective mapping from $\{(c(v), c(w)) : (v, w) \in M\}$ onto D.

See also Figure 1(b). In what follows, we never consider non-coloured dominoes and/or tilings, and hence drop "coloured" from now on.

We can distinguish domino tilings by whether they are using all given dominoes. In the Exact Domino Tiling problem every domino must be used exactly once in the tiling, so the number of dominoes must equal half the number of vertices of the layout graph. In the Partial Domino Tiling problem we can have arbitrarily many dominoes. Watson and Worman [5] showed the following results:

- Exact Domino Tiling is solvable if the layout graph is a path or a cycle.
- Partial Domino Tiling is NP-hard, even if the layout graph is a tree.

*School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada, e-mail biedl@uwaterloo.ca. Research partially supported by NSERC. The author would like to thank Jan Kratochvíl, Mark Watson and Chris Worman for helpful discussions.

Watson and Worman left as an open question whether Exact Domino Tiling is NP-hard. We prove this, and strengthen their NP-hardness results, as follows:

- We show that Exact Domino Tiling is NP-hard, even if the layout graph is a *caterpillar*, i.e., a tree that consists of a path with degree-1 vertices attached.
- We show that Partial Domino Tiling is NP-hard, even if the layout graph is a cycle or a path.

Both our NP-hardness proofs and the one in [5] use an unbounded number of colours. This is unrealistic, since normally dominoes are "coloured" with between 0 and 9 dots. We hence next study dominoes with few colours. We show that coloured domino tiling is NP-hard even if only 3 colours are used for more complicated layouts.

We are generally only concerned with proving NP-hardness; the domino tiling problems are clearly verifiable in polynomial time, and hence our results really prove NP-completeness.

2 Paths, Cycles, Trees

Domino tilings can also be described in terms of graph homomorphisms. We need some definitions first. Assume we are given a layout graph G^L and a perfect matching M in it. The *contracted graph* $G^C(M)$ results from contracting every edge that is not in M. For cycles and trees, all perfect matchings give the same contracted graph, and we write G^C instead of $G^C(M)$.

A multi-set of dominoes can conveniently be expressed via the *domino graph*, which is a multi-graph that has a vertex for every colour, and for every domino (c_i, c_j) an edge between the vertices for c_i and c_j.

A *graph homomorphism* from graph G to graph H is a mapping from $V(G)$ to $V(H)$ such that edges are mapped to edges. It is *vertex-injective* if no two vertices in G are mapped to the same vertex of H, and *edge-injective* if the mapping of the edges is injective. See [2] for more on graph homomorphisms. For domino tiling, we are interested in edge-injective (but not necessarily vertex-injective) homomorphisms. To our knowledge, this topic has not been studied before.

The equivalence of edge-injective homomorphisms and domino tiling follows from translating the condition on the colouring function of a domino tiling to the mapping function of the homomorphism. We omit the details here for space reasons.

Observation 1 *A layout L has a domino tiling if and only if the layout graph G^L has a perfect matching M such that $G^C(M)$ has an edge-injective homomorphism to the domino graph.*

For our NP-hardness results, we now study the complexity of edge-injective homomorphisms.

Theorem 1 *Testing whether G has an edge-injective homomorphism to H is NP-hard, even if*

(a) G is a cycle, or

(b) G is a path, or

(c) G is a caterpillar and G and H have equally many edges.

Proof. (Sketch) The reduction in all cases is from Hamiltonian Cycle or Hamiltonian Path: Given a 3-regular graph H with n vertices, does it have a cycle/path that visits every vertex exactly once? This is known to be NP-hard [1].

For (a), note that H has a Hamiltonian cycle if and only if a cycle with n vertices has an edge-injective homomorphism to H. (Since H has maximum degree 3, edge-injective implies vertex-injective.)

For (b), let G be a path with $n+2$ vertices $v_0, v_1, \dots, v_n, v_{n+1}$. For any edge-injective homomorphism, the images of $v_1, \dots, v_n$ must be distinct, again since H has maximum degree 3, so this gives a Hamiltonian path of H. Conversely, any Hamiltonian path of H can easily be extended into a path of length $n+2$ since H is 3-regular, so then G has an edge-injective homomorphism to H.

The reduction for (c) is the most complicated. Assume H' is a 3-regular graph with n vertices in which we are searching for a Hamiltonian path. Graph H is obtained from H' by subdividing all edges. H' has $\frac{3}{2}n$ edges, so H has $3n$ edges.

Graph G consists of a *backbone* $b_1, a_1, b_2, \dots, a_{n-1}, b_n$ and $n+2$ *legs* attached to $b_1, \dots, b_n$ such that each b_i has degree 3. See Figure 2. Clearly G has $3n$ edges, so $|E(G)| = |E(H)|$ as desired. Also note that G is a caterpillar.

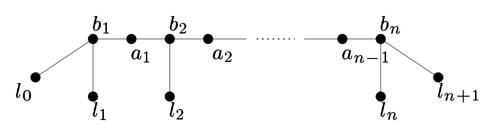

Figure 2: The graph G used for the reduction.

Since H' is 3-regular, any edge-injective homomorphism of G to H must map vertices b_i to original vertices of H', and vertices a_i to subdivision vertices of H, i.e., edges of H'. The image of the backbone hence yields a path in H', and it must be Hamiltonian since H has maximum degree 3. On the other hand, if H' has a Hamiltonian path, then we map the backbone to the corresponding path in H. The legs are mapped to subdivision vertices of edges in H' that are not in the Hamiltonian path, and hence use up all the remaining dominoes. □

Now we are ready for our first NP-hardness results for domino tiling.

Theorem 2 PARTIAL DOMINO TILING *is NP-hard, even if the layout graph is a path or a cycle.* EXACT DOMINO TILING *is NP-hard, even if the layout graph is a caterpillar.*

Proof. By Theorem 1 and Observation 1, all that remains to do is to show that the path/cycle/caterpillar is indeed the contracted graph of a layout graph. This is trivial for a path and for a cycle with at least 4 vertices; for the caterpillar we show the layout in Figure 3. □

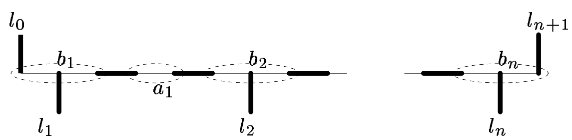

Figure 3: A layout graph that is a caterpillar, and contracting every edge not in the unique perfect matching (bold) gives graph G from Figure 2.

3 Few colours

We now study the more realistic case of dominoes with a small number of colours. As our main result, we show that the problem then still remains NP-hard, even with only 3 colours.

Note that a constant number of colours corresponds to a target graph H (for the edge-injective homomorphism) that has a constant number of vertices. It is well-known that testing the existence of a graph homomorphism from G to H is NP-hard even if H is a 3-cycle. This does not immediately imply NP-hardness for domino tiling for two reasons: (1) We must restrict graph G to be a graph that is the contracted graph of some layout graph, regardless of how the perfect matching is chosen. (2) The layout graph must in fact be the the graph of a layout, i.e., a subgraph of the rectangular grid. To ease presentation, we defer this part to later, and first show NP-hardness of coloured domino tiling for graphs that need not be graphs of layouts.[1]

The reduction, as for graph homomorphism, is from *3-colouring*, i.e., given a graph G, can the vertices be coloured with 3 colours such that no two endpoints of an edge have the same colour? This remains NP-hard even if every vertex of G has at least three incident edges [1]. We create the layout graph G^L from G by modifying the vicinity of every vertex v. We first replace v by a path of length $\deg(v)$, where each vertex of the path is incident to one of the edges of v. Then we subdivide each edge of the path and add a vertex of degree 1 on it. See Figure 4.

For space reasons, we omit the (simple) proofs of this and all following claims.

[1] Note that the definition of coloured domino tiling does not require any geometry of a layout.

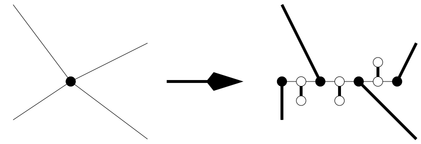

Figure 4: Replacing each vertex by a path. and the unique perfect matching (bold).

Claim 1 G^L *has a unique perfect matching.*

Claim 2 G^C *is 3-colourable if and only if* G *is 3-colourable.*

Theorem 3 *Testing whether a graph (not necessarily resulting from a layout) can be tiled with a given set of dominoes is NP-hard, even if only 3 colours are used by the dominoes.*

Proof. Let G be a graph for which we want to test the existence of a 3-colouring. Define the graph G^L as described above. Let D be a set of dominoes with three colours c_1, c_2, c_3, and for each $c_i \neq c_j$, add sufficiently many dominoes (c_i, c_j).[2] Assume G^L has a domino tiling, with colouring function c. Any vertex v in G^C is obtained by contracting some vertices $w_1, \ldots, w_k$ of G^L that are connected by edges not in the perfect matching M. Thus $w_1, \ldots, w_k$ all must have the same colour; set $c(v) = c(w_1)$. Any edge in G^C corresponds to a domino in the tiling; since there are no uni-coloured dominoes therefore this colouring of G^C is a 3-colouring of G^C. By Claim 2 G is 3-colourable. The other direction is similar. If G is 3-colourable, then so is G^C. Assign to each vertex in G^L the colour of the vertex in G^C into which it was contracted. Then all matching edges have differently coloured endpoints, and thus correspond to a domino. This gives a valid domino tiling, since every type of domino exists sufficiently often. □

Now we extend Theorem 3 to graphs that result from a layout. The basic idea is the same, but we need to modify our graph further and apply some graph drawing results to find the layout.

Recall that G was the graph which we wanted to 3-colour. We now choose G to be a *planar graph*, i.e., G can be drawn without crossing in the plane. Furthermore, we assume that G has maximum degree 4. It is known that 3-colouring remains NP-hard even for planar graphs with maximum degree 4 [1].

Now we create a *planar orthogonal drawing* of G, i.e., a crossing-free drawing of G on the 2D rectangular grid such that every edge is routed as a sequence of horizontal and vertical line segments. Such drawings exist

[2] "Sufficiently many" means "enough copies such that we can never run out of dominoes." More precisely, if the layout graph has $2n$ vertices, then n dominoes are needed in the tiling; adding n dominoes of every kind is hence sufficient.

for any planar graph with maximum degree 4, see for example [3]. Furthermore, the total edge-length is polynomial.

We obtain the layout by expanding the planar orthogonal drawing of G. To do so, we first scale the drawing, i.e., we replace each row/column of the grid by many rows/columns (20 should be enough), to achieve sufficiently much separation between parallel edge segments. Then we replace each vertex, line segment, and *bend* (place where an edge changes direction) with one of the gadgets in Figure 5.

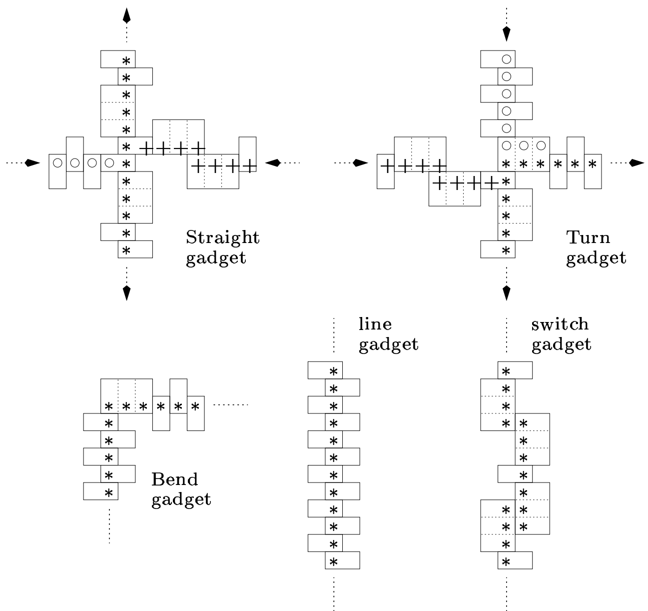

Figure 5: Gadgets used in the reduction.

Claim 3 *Assume there are no uni-coloured dominoes. Then any gadget has only one perfect matching (the one shown in Figure 5) that could be used in a coloured domino tiling.*

Claim 4 *Assume we have a domino tiling of the gadgets without uni-coloured dominoes. Then within any gadget, all squares marked with the same symbol in Figure 5 must have the same colour. Moreover, squares marked $*$ have a different colour than squares marked $+$ or $\circ$. Squares marked $+$ and $\circ$ can, but need not, have the same colour.*

Direct the edges of G such that every vertex has at most two outgoing edges and at most two incoming edges. (For example, make G Eulerian by adding edges between vertices of odd degree, direct all edges while walking along an Eulerian circuit, and then delete the added edges.) Now replace the drawing by gadgets as follows. Replace each bend with the bend-gadget, rotated if needed such that the lines marked with $*$ coincide with the drawing of the incident edge segments. Each vertex v is replaced with either the straight-gadget or the turn-gadget, rotated if needed, such that outgoing edges of v are on lines marked with $*$. Finally add line gadgets to replace horizontal or vertical line segments; note that a line gadget can be made arbitrarily long. However, inserting a line gadget may lead to a 2×2-square where it attaches to a bend-gadget or vertex-gadget. In this case, replace part of the line gadget by the switch gadget, which changes how dominoes attach (relative to the "line"), and hence avoids creation of a 2×2-square.

This finishes the description of our layout L. The dominoes D have three colours c_1, c_2, c_3, and for each $c_i \neq c_j$, we have sufficiently many dominoes (c_i, c_j). Note that we have no uni-coloured dominoes, so Claims 3 and 4 hold and we need to consider only one perfect matching and contracted graph G^C. This contracted graph contains graph G as an induced subgraph, hence if G^C has an edge-injective homomorphism into a 3-cycle, then G can be 3-coloured as desired.

On the other hand, given a 3-coloring of G, we can obtain a domino tiling by letting $*$ at the gadget of v be the colour of v; this requires no uni-coloured dominoes and is a domino tiling since we have sufficiently many dominoes of all types.

Finally, note that since the drawing of G had polynomial total edge length, the construction is polynomial in the size of G, which finishes the NP-hardness proof.

Theorem 4 PARTIAL DOMINO TILING *is NP-hard, even if only 3 colours are used.*

We can extend this result to EXACT DOMINO TILING, by using three copies of the previous layout and adding a connection gadget.

Corollary 5 EXACT DOMINO TILING *is NP-hard, even if only 3 colours are used.*

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