Computing bitangents for ellipses

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Abstract

We show how to compute the bitangents of two ellipses, and how to evaluate the predicates required to compute objects like convex hulls or visibility complexes for ellipses. Also, we show how to compute the tangency points of the bitangents.

1 Introduction

1.1 Motivations

Bitangents of planar objects are the basic component of many geometric objects like convex hulls, pseudo-triangulations and visibility complexes.

Many algorithms [AP03, HP04] dealing with bitangents of convex objects need only two operations :

- Given two objects, and a *type* (defined in 1.2.2), compute their bitangent which has this type.
- Given two directed bitangents, decide whether their angle is counter-clockwise directed (this is called the χ_2 predicate).

In this paper, we solve these two issues for ellipses, in order to be able to use the existing algorithms with them.

1.2 Definitions and notations

1.2.1 Ellipses

We work with two non degenerate ellipses defined by a degree 2 bivariate polynomial with integer coefficients :

$$\begin{array}{rcrcl} E & : & aX^2 + bXY + cY^2 + dX + eY + f & = & 0 \\ E' & : & a'X^2 + b'XY + c'Y^2 + d'X + e'Y + f' & = & 0 \end{array}$$

1.2.2 The type of a bitangent

Bitangents are regarded as directed from E towards E'. Then, for each bitangent u, E and E' are to the left or to the right of u. The *type* of u is the pair in $\{L, R\}^2$ whose first (respectively second) letter tells whether E (respectively E') is to the left (L) or right (R) of u. For instance the outer bitangents have type LL and RR, while the separating bitangents have type LR and RL, as shown on figure 1.

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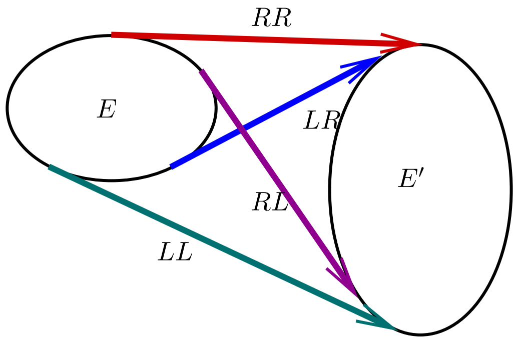

Figure 1: The types of the bitangents

Lemma 1 *When the ellipses do not intersect and the one is not contained in the other, there is a one to one correspondence between the four bitangents and the four types. The cyclic ordering of the directions of the bitangents is LL, LR, RL, RR.*

We will assume that E and E' do not intersect, and that one of them is not contained in the other.

1.3 Previous works

The case of circles with rational radius has been dealt with by Pierre Angelier [Ang02].

In parallel with us, I. Emiris, E. Tsigaridas and G. Tsoumas [ETT05] have worked on bitangents of ellipses, with a different application in mind : to compute the Voronoi diagram of a set of ellipses. When compared with our method, their method does not allow to compute the types of the bitangents, and needs to compute both coefficients of the cartesian equation of the bitangents.

2 Computing the slope of the bitangents

2.1 Tangents

The vertical bitangents, if any, are easy to compute. Therefore, we will focus on the non-vertical bitangents. Let L be a non-vertical line, with equation $Y = UX + V$. L is tangent to E if and only if it intersects E in a single point.

Substituting $UX + V$ to Y in the equation of E yields a univariate degree 2 polynomial in X, which has a single root if and only if its discriminant is null, that is :

$$t_1U^2 + t_2UV + t_3V^2 + t_4U + t_5V + t_6 = 0$$

with

$$\begin{array}{rclrcl} t_1 & = & e^2 - 4cf & t_4 & = & 2de - 4bf \\ t_2 & = & 4dc - 2be & t_5 & = & 2db - 4ae \\ t_3 & = & b^2 - 4ac & t_6 & = & d^2 - 4af \end{array}$$

2.2 The polynomial system

A similar equation can be worked out for E', so that L is tangent to E and E' if and only if

$$\left\{ \begin{array}{l} t_1U^2 + t_2UV + t_3V^2 + t_4U + t_5V + t_6 = 0 \\ \text{and} \\ t'_1U^2 + t'_2UV + t'_3V^2 + t'_4U + t'_5V + t'_6 = 0 \end{array} \right.$$

2.3 Elimination

To solve this system, we can eliminate V, and obtain that L is tangent to E and E' if and only if :

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \alpha U + \beta = 0 = \gamma U^2 + \delta U + \zeta \\ \text{and} \\ t_1U^2 + t_2UV + t_3V^2 + t_4U + t_5V + t_6 = 0 \end{array} \right. \\ \text{or} \\ \left\{ \begin{array}{l} V = -(\gamma U^2 + \delta U + \zeta)/(\alpha U + \beta) \\ \text{and} \\ \Pi(U) = 0 \end{array} \right. \end{array} \right.$$

with :

$$\begin{array}{rclrcl} \alpha & = & t'_3t_2 - t_3t'_2 & \gamma & = & t'_3t_1 - t_3t'_1 \\ \beta & = & t'_3t_5 - t_3t'_5 & \delta & = & t'_3t_4 - t_3t'_4 \\ & & & \zeta & = & t'_3t_6 - t_3t'_6 \end{array}$$

and

$$\begin{array}{rcl} \Pi(U) & = & \lambda_4U^4 + \lambda_3U^3 + \lambda_2U^2 + \lambda_1U + \lambda_0 \\ \lambda_4 & = & t_1\alpha^2 + t_3\gamma^2 - t_2\gamma\alpha \\ \lambda_3 & = & -t_2\gamma\beta - t_2\delta\alpha + \alpha^2t_4 + 2t_1\alpha\beta + 2t_3\gamma\delta - t_5\gamma\alpha \\ \lambda_2 & = & 2\alpha\beta t_4 + t_3\delta^2 - t_5\gamma\beta - t_2\zeta\alpha + \alpha^2t_6 - t_5\delta\alpha \\ & & +t_1\beta^2 - t_2\delta\beta + 2t_3\gamma\zeta \\ \lambda_1 & = & -t_2\zeta\beta - t_5\zeta\alpha + 2t_3\delta\zeta + 2\alpha\beta t_6 + \beta^2t_4 - t_5\delta\beta \\ \lambda_0 & = & -t_5\zeta\beta + t_3\zeta^2 + \beta^2t_6 \end{array}$$

Two simple observations :

- If there exists U such that $\alpha U + \beta = 0$ and $\gamma U^2 + \delta U + \zeta = 0$, we get two different (since the ellipse is not reduced to a point) values of V solution for a same U. This means that there are two parallel bitangents. The converse also holds.
- If U is such that $\alpha U + \beta \neq 0$, then there is a bitangent with slope U if and only if $\Pi(U) = 0$.

Moreover, using some perturbation arguments, it is possible to prove that :

- there exists a bitangent with slope U if and only if U is a root of Π
- the multiplicity of a root U of Π is the number of bitangents with slope U
- the degree of Π is 4 when no bitangent is vertical, 3 when one bitangent is vertical, and 2 when two bitangents are vertical.

The coefficients of Π are polynomials in the coefficients of E and E'. Their degree appears to be 10, however, it turns out that the higher degree terms cancel out, so that the degree of all the λ_i is only 8.

3 Studying V

Let L be a non-vertical bitangent, with equation $y = Ux + V$. In this section, we show how to decide whether L is above or below E.

Recall that :

$$\begin{array}{rcl} t_1U^2 + t_2UV + t_3V^2 + t_4U + t_5V + t_6 & = & 0 \\ (\alpha U + \beta)V + \gamma U^2 + \delta U + \zeta & = & 0 \end{array}$$

If $\alpha U + \beta = 0$, then there are two bitangents with slope U, the one is below, the other is above.

Otherwise,

$$V = -\frac{\gamma U^2 + \delta U + \zeta}{\alpha U + \beta}$$

We also know that

$$V = \frac{-t_2U - t_5 + \epsilon\sqrt{(t_2U - t_5)^2 - 4t_3(t_1U^2 + t_4U + t_6)}}{2t_3}$$

where $\epsilon \in \{1, -1\}$, so that $\frac{-t_2U - t_5}{2t_3}$ is in between the two possible values for V.

Therefore, comparing $-\frac{\gamma U^2 + \delta U + \zeta}{\alpha U + \beta}$ with $\frac{-t_2U - t_5}{2t_3}$ solves the matter. The difference of these values is the quotient of a degree 2 polynomial in U by a degree 1 polynomial in U. The signs of these polynomials evaluated at U can be computed using Sturm sequences, and yield the sign of the difference.

4 Identifying the bitangent that we want

In this section, we show how to find out the types of the bitangents whose slopes are the roots of Π.

It would be possible to examine each root in turn, and, determine the relative position of both ellipses with respect to it, but this would require painful algebraic computations, while there is a much nicer solution.

Indeed, we know for every bitangent whether it is above or below E, so that we can sort them in cyclic order around E. Then, if we know the type of one of

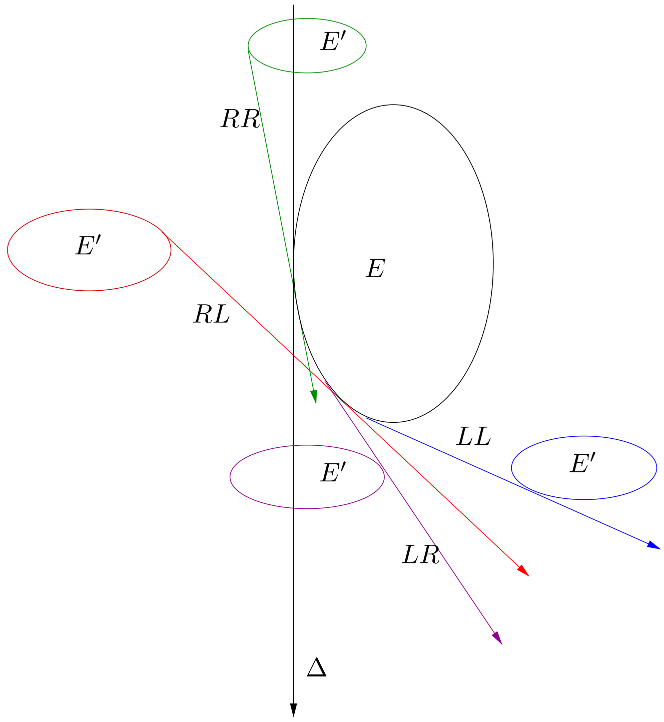

Figure 2: Relative position of Δ and E' : the four possible cases, with, in each case, the bitangent that succeeds to Δ.

them, we can infer the type of all the others (recall lemma 1).

Now, how do we come to know the type of one of the bitangents?

Let Δ be a tangent to E, directed so that E is to the left of Δ. We can locate Δ in the cyclic order of the bitangents, and find the bitangent B that is the successor of Δ. It turns out that the relative position of Δ and E' allows to infer the type of B. For instance, if E' is to the left of Δ, then B has type LL. The four possible cases are illustrated on figure 2.

In practice, we choose an easy to compute Δ : the left-hand side vertical tangent to E. Then we decide whether E' lies entirely to the left or to the right of Δ or intersects Δ above or below E. This can be done by comparing degree 2 algebraic numbers.

5 Computing the tangency points

5.1 The polynomial system

The tangency points are harder to compute. Here is a sketch of a method.

If (X,Y) is a point of E, a parametric equation of the tangent to E running through (X,Y) is :

$$\begin{aligned} x &= X - t(2cY + bX + e) \\ y &= Y + t(2aX + bY + d) \end{aligned}$$

with $t \in \mathbb{R}$.

Again, in the equation of E', we can substitute to (X,Y) the values of (x,y) above. Thus, we obtain a univariate degree 2 polynomial (in t). The roots of this polynomial correspond to the intersections of the tangent with E'. This tangent is a bitangent if and only if the discriminant of the latter polynomial is null. The discriminant is a bivariate polynomial in X and Y. This time, it has degree 4, and is too large to reproduce here. Let us denote it $\Delta(X,Y)$.

We obtained that (X,Y) is a tangency point upon E of a bitangent to E and E' if and only if :

$$\left\{ \begin{array}{l} \Delta(X,Y) = 0 \\ \text{and} \\ aX^2 + bXY + cY^2 + dX + eY + f = 0 \end{array} \right.$$

5.2 Solving the system

Eliminating X or Y from this system leads to computations too large to be carried out by hand. It is nevertheless possible to perform them using a Gröbner basis computation system like [JCF]. The result is a degree 4 polynomial in X or Y, whose coefficients are degree 10 polynomials in the coefficients of E and E'. Those latter polynomials are really huge (over 2000 monomials), probably causing any implementation to be impractical.

5.3 Coordinate system change

In fact, there is a case where the elimination is greatly simplified : when b, d and e are null. Figure 3 presents a maple session that performs the elimination of Y from the system, once the equation of E is simplified. The polynomial obtained is still large, but far more reasonable. It is a degree 4 polynomial, whose coefficients are degree 8 polynomials in the coefficients of the ellipses.

It happens to be possible to reduce the general problem to this simplified case, by means of a coordinate system change. First, the following transformation cancels b[1] :

$$(X,Y) \rightarrow (X, Y + \frac{b}{2c}X)$$

and, subsequently, a translation allows to cancel d and e.

Now, given E and F with general equations, we can perform the transformation, compute the x-coordinates of the tangency points in the transformed scene, and retrieve the x-coordinates of the tangency points in the original scene, by a rational translation. The y-coordinate can be computed similarly. The types of the bitangents can be computed in essentially the same way as in section 4.

[1] using a rotation might appear more natural, but, both coordinates would be altered, and, in general, the coefficients of the matrix of the rotation will not be rational

```
E1:=a1*X^2+c1*Y^2+f1;
E2:=a2*x^2+b2*x*y+c2*y^2+d2*x+e2*y+f2;

tg2:=subs(x=X-t*c1*Y,y=Y+t*a1*X,E2);

tg2e:=collect(expand(tg2),t);
tg20:=coeff(tg2e,t,0);
tg21:=coeff(tg2e,t,1);
tg22:=coeff(tg2e,t,2);

delta:=collect(expand(tg21^2-4*tg20*tg22),
   [Y,X],recursive);
# Eliminate Y from delta :
# - decompose delta as dP+Y*dQ where dP and dQ
# contain only even powers of Y and substitute
# z to Y^2
# - multiply by dP-Y*dQ -> dS
# - eliminate z from dS using E1
dP:=coeff(delta,Y,4)*z^2+
  coeff(delta,Y,2)*z+coeff(delta,Y,0);
dQ:=coeff(delta,Y,3)*z+coeff(delta,Y,1);
dS:=collect(expand(dP^2-z*dQ^2),
  [X,z],distributed);
P:=collect(simplify(
   subs(z=(-f1-a1*X^2)/c1,dS)),X);
```

Figure 3: A maple session to compute a cancelling polynomial of the x-coordinates of the tangency points

There is, however, a drawback : the transformation multiplies by 16 the bit complexity of the coefficients of the ellipses.

6 Conclusion

6.1 Evaluating χ_2

The evaluation of the χ_2 predicate over two directed bitangents essentially boils down to comparing their slopes, that is, comparing two degree 4 algebraic numbers, which can be performed using Sturm sequences.

6.2 Implementation

We have implemented the method described above. To handle the degree 4 algebraic numbers, we use I. Emiris' and E. Tsigaridas' "RootOf" package [ET04].

A preliminary benchmark has been performed on a Pentium IV-2400. The coefficients of the ellipses have from 8 to 16 decimal digits (they are computed from mouse clicks, hence the variability, in fact the coefficients of X^2, XY and Y^2 are smaller than the other three). In those conditions, computing a bitangent costs somewhere around 5 milliseconds.

6.3 Future work

Currently, the *Visibility Complex* package [Ang02] uses another predicate in the presence of constraints. This predicate decides whether a tangency point of a bitangent lies above or below another bitangent. To decide this predicate for ellipses, one needs to evaluate the sign of a bivariate polynomial over degree 4 algebraic numbers from different extensions, which is far more costly than comparing slopes. Therefore, we intend to modify the algorithm used in order to drop the need for this predicate.

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