Matching Edges and Faces in Polygonal Partitions

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Abstract

We define general Laman (count) conditions for edges and faces of polygonal partitions in the plane. Several well-known classes, including k-regular partitions, k-angulations, and rank-k pseudo-triangulations, are shown to fulfill such conditions. As a consequence, non-trivial perfect matchings exist between the edge sets (or face sets) of two such structures when they live on the same point set. We also describe a link to spanning tree decompositions that applies to quadrangulations and certain pseudo-triangulations.

1 Introduction

There exist several results [2] concerning matchings between the edges (or triangles) in two given triangulations on top of the same point set S. For example, for any two triangulations T_1 and T_2 of S, we can pair each edge $e_1 \in T_1$ with an edge $e_2 \in T_2$ such that either $e_1 = e_2$ or e_1 crosses e_2. Moreover, each triangle $\Delta_1 \in T_1$ can be paired with a triangle $\Delta_2 \in T_2$ such that either $\Delta_1 = \Delta_2$ or Δ_1 partially overlaps with Δ_2. Perfect matchings of this kind prove useful for obtaining lower bounds on the edge length of the minimum weight triangulation of S; see [2].

Unfortunately, pseudo-triangulations (see Section 3 for a definition) do not share these properties. Figure 1 depicts two pseudo-triangulations PT_1 (left) and PT_2 (right) on a set of five points. Note that PT_1 and PT_2 have the same number of edges (and faces). The bold edge in PT_1 neither crosses, nor coincides with, an edge in PT_2. Thus no edge matching as above is possible. Also, the two shaded faces in PT_2 both overlap only with the shaded face in PT_1. This rules out a face matching.

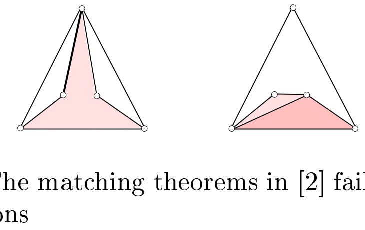

Figure 1: The matching theorems in [2] fail for pseudo-triangulations

We intend to show that perfect machings can be retained when 'crossing' and 'overlap', respectively, is relaxed to vertex incidence. In fact, such incidence matchings also exist for polygonal partitions different from pseudo-triangulations. We define a general condition that guarantees the existence of incidence matchings for edges and faces in two polygonal partitions with the same vertex set. This condition (sometimes) also implies decomposability into edge-disjoint spanning trees.

2 Generalized Laman property

Throughout, let S be a finite set of (at least three) points in the plane. Let $conv(S)$ denote the convex hull of S. A *polygonal partition*, P, *on* S is a partition of $conv(S)$ into simple polygons (faces) such that S is the vertex set of P, and such that each edge of P which is not an edge of $conv(S)$ is common to exactly two faces.

Let now P be any polygonal partition on S. Throughout, let the term 'object' consistently stand for either 'edge' or 'face'. Consider an arbitrary subset $S' \subseteq S$. We say that an object x of P is *spanned* by S' if x has all its incident vertices in S'. Denote with $\alpha(S')$ the number of objects of P that are spanned by S'. Further, let $n(S')$ be the cardinality of S', and let $h(S')$ be the number of vertices of $conv(S')$. Note that $\alpha(S)$ expresses the total number of objects of P. As P defines a planar straight line graph on S, $\alpha(S)$ is a linear function of $n(S)$. We call P *object-Laman* if there exist three constants $c_1 \geq c_2 \geq 0$ and $c_3 \geq -1$ such that the following two conditions hold:

$$\alpha(S) = c_1 n(S) - c_2 h(S) - c_3$$

and, for each subset $S' \subset S$ with $n(S') \geq 2$,

$$\alpha(S') \leq c_1 n(S') - c_2 h(S') - c_3$$

the so-called *hereditary Laman condition*. We term the triple (c_1, c_2, c_3) the *(object) characteristic* of P. Classical planar Laman graphs [10] have embeddings as straight line graphs that yield polygonal partitions with edge characteristic $(2, 0, 3)$; see [8]. That is, a Laman graph on n vertices has precisely $2n - 3$ edges, and each subgraph on $n' \geq 2$ vertices has at most $2n' - 3$ edges. In [3], the concept of bounded graph density from [10] is extended to general functions of n. Dealing with purely graph-theoretical concepts, they do not consider the number of convex hull points as a parameter.

An object x of P is said to be *covered* by a subset $S' \subseteq S$ if x has at least one incident vertex in S'. Let $\beta(S')$ denote the number of objects of P that are covered by S'. Clearly $\beta(S') \geq \alpha(S')$ holds, as each object spanned by S' is also covered by S'. Polygonal partitions that are object-Laman satisfy the following property. (We omit most proofs due to lack of space.)

Lemma 1 *Let P be any polygonal partition on S that is object-Laman with characteristic $(c_1, c_2, c_3 \geq 0)$. Then $\beta(S') \geq c_1 n(S') - c_2 h(S') - c_3$ holds, for each $S' \subseteq S$.*

The object Laman property is strong enough to imply a non-trivial bijection between the edge sets (or face sets) of two polygonal partitions that live on the same configuration of points.

Theorem 2 *Let S be a finite set of points in the plane. Let P_1 and P_2 be any two polygonal partitions on S that are object-Laman with same characteristic $(c_1, c_2, c_3 \geq 0)$. There exists a perfect matching between the set of objects of P_1 and the set of objects of P_2 such that matched objects share a vertex.*

Proof. Let O_i be the set of objects of P_i, for $i = 1, 2$. For a subset $X \subseteq O_1$, let $Y \subseteq O_2$ denote the set of objects that possibly can be matched to some object in X. More precisely, Y contains all objects $y \in O_2$ such that y shares some vertex with an object in O_1. We show $|Y| \geq |X|$. That is, the Hall condition [5] for the marriage theorem is fulfilled, which implies the existence of a perfect matching between O_1 and O_2.

Let S' be the subset of S that consists of all the vertices of the objects in X. That is, X is the set of objects of P_1 that are spanned by S'. If $n(S') \leq 1$ then $|X| = 0$, and $|Y| \geq |X|$ clearly holds. Let $n(S') \geq 2$. By the assumed Laman property for P_1 we have $|X| \leq c_1 n(S') - c_2 h(S') - c_3$. On the other hand, Y is precisely the set of objects of P_2 that are covered by S'. By the assumed Laman property for P_2 we now get $|Y| \geq c_1 n(S') - c_2 h(S') - c_3$ from Lemma 1. We conclude $|Y| \geq |X|$ again. $\square$

The Eulerian relation for planar graphs implies a correspondence between the edge-Laman and the face-Laman property. From now on, let us write the number $\alpha(S')$ of objects spanned by a subset $S' \subset S$ as $e(S')$ if the objects are edges, and as $f(S')$ if the objects are faces.

Lemma 3 *Let a polygonal partition P on S be given and assume that P is edge-Laman with characteristic $(c_1 \geq 1, c_2 \leq c_1 - 1, c_3 \geq 1)$. Then P is face-Laman with characteristic $(c_1 - 1, c_2, c_3 - 1)$.*

3 Some relevant polygonal partitions

The edge-Laman and the face-Laman property are quite natural; they are shared by several well-known classes of polygonal partitions. In the sequel, we require $n(S') \geq 2$ for the considered subset $S' \subset S$. This ensures that the formulas below yield nonnegative values for $e(S')$ and $f(S')$. Let us denote with $A(S')$ the subset of objects (under consideration) spanned by S'.

3.1 Pseudo-triangulations

A *pseudo-triangulation*, PT, of S is a polygonal partition on S whose faces are pseudo-triangles, i.e., polygons with exactly three convex vertices. A vertex of PT is called *pointed* if its incident edges span a convex angle. Let PT contain exactly p pointed vertices. In [1], the *(edge) rank* of PT is defined as $n(S) - p$, the number of non-pointed vertices. The maximum rank of PT is $n(S) - h(S)$, in which case PT is a triangulation. The minimum rank of PT is zero, and PT is commonly called a *pointed* (or *minimum*) pseudo-triangulation in that case.

It is well known that every rank-k pseudo-triangulation of S has exactly $e(S) = 2n(S) + k - 3$ edges. Consider a subset $S' \subseteq S$, and assume that the set $A(S')$ defines a pseudo-triangulation of S'. As each vertex that is non-pointed in $A(S')$ has to be non-pointed in PT as well, the rank of $A(S')$ is at most k. On the other hand, if $A(S')$ is a proper subset of a pseudo-triangulation of S', then $A(S')$ can be completed to one with rank k. This shows $e(S') \leq 2n(S') + k - 3$. That is, the hereditary Laman condition is fulfilled. We conclude that PT is edge-Laman, provided that $k \leq 4$. In conjunction with Lemma 3 we obtain:

Observation 1 *For $k \leq 4$, every rank-k pseudo-triangulation of S is edge-Laman with characteristic $(2, 0, 3 - k)$. For $k \leq 2$, every rank-k pseudo-triangulation of S is face-Laman with characteristic $(1, 0, 2 - k)$.*

It has been known [14] that pointed pseudo-triangulations enjoy the edge Laman property; in fact, they are planar Laman graphs in the classical sense [8]. A similar egde Laman condition for general pseudo-triangulations is used in [12] to define their combinatorial abstractions. In Subsection 3.2 we will observe that triangulations are both edge-Laman and face-Laman. Pseudo-triangulations of arbitrary rank share neither property, in general.

3.2 k-angulations

A *k-angulation* of S, $k \geq 3$, is a polygonal partition on S all whose faces are k-gons, i.e., polygons with exactly k vertices. Prominent representatives are triangulations ($k = 3$) and quadrangulations ($k = 4$). Note that we do not require convexity of the faces. It is well known that every triangulation of S contains the same number of edges and triangles. This fact generalizes to k-angulations, for $k \geq 4$.

The sum of angles in any k-gon is $\pi(k-2)$. The sum of angles in all the faces of a k-angulation, Q, of S thus is $\pi(h(S)-2)$ for angles at vertices of $conv(S)$ plus $2\pi(n(S)-h(S))$ for angles at vertices interior to $conv(S)$. Dividing by $\pi(k-2)$ gives the number of Q's faces,

$$f(S) = \frac{2n(S) - h(S) - 2}{k-2}. \qquad (1)$$

Respecting the exterior face, the Eulerian relation gives $n(S) - e(S) + (f(S) + 1) = 2$. We plug in (1) and get the number of edges of Q,

$$e(S) = \frac{kn(S) - h(S) - k}{k-2}. \qquad (2)$$

Consider a subset $S' \subseteq S$. If the set $A(S')$ is a k-angulation of S' then (2) holds with S replaced by S'. But this formula also describes the maximum number of possible edges when k-gons on top of S' are constructed. Therefore, the hereditary Laman condition is fulfilled. Together with Lemma 3 this yields:

Observation 2 *Every k-angulation of S, $k \geq 3$, is object-Laman with edge characteristic $\frac{1}{k-2}(k,1,k)$ and face characteristic $\frac{1}{k-2}(2,1,2)$.*

3.3 k-regular partitions

A polygonal partition P is called *k-regular* if the degree of every vertex of P is exactly k. For $k = 3$, simple partitions (in the classical sense) are obtained. For instance, Schlegel diagrams [6] of simple three-dimensional polytopes, and thus power diagrams and Voronoi diagrams [4] in suitable domains, belong to this class. Apart from trivial cases, k-regular partitions only exist for $3 \leq k \leq 5$.

Let now P be a k-regular partition on S. Each vertex of P is incident to exactly k edges, and each edge of P has two vertices. Consequently,

$$e(S) = \frac{k}{2} n(S). \qquad (3)$$

Applying the Eulerian formula gives

$$f(S) = (\frac{k}{2} - 1)n(S) + 1. \qquad (4)$$

Observe that (3) is also the maximum number of possible edges when drawing on top of S a planar straight line graph with vertex degree at most k. But, for any $S' \subseteq S$, each vertex in the set $A(S')$ is of degree at most k, which shows that the hereditary Laman condition holds for P's edges.

In the edge characteristic of P, the constant c_3 is zero, and Lemma 3 does not apply. However, by using the arguments above on (4), P is easily seen to fulfill the hereditary Laman condition for faces, too. We summarize:

Observation 3 *Every k-regular polygonal partition on S, $3 \leq k \leq 5$, is object-Laman with edge characteristic $(\frac{k}{2}, 0, 0)$ and face characteristic $(\frac{k}{2} - 1, 0, -1)$.*

For straight line graphs on S (as opposed to polygonal partitions on S) the notion of k-regularity is meaningful for general k. For example, for $k = 2$ we obtain vertex-disjoint covering cycles, and for $k = 1$ we obtain perfect matchings. It follows that these structures are edge-Laman with characteristics $(1, 0, 0)$ and $(\frac{1}{2}, 0, 0)$, respectively. Finally, note that any spanning tree of S is edge-Laman with characteristic $(1, 0, 1)$.

4 Incidence matching for edges and faces

Our results in Section 3 combine with Theorem 2 (the incidence matching theorem) in the following way.

Theorem 4 *Let S be a finite set of points in the plane. Let P and Q be two structures on top of S, from one of the following classes (k fixed): Rank-k pseudo-triangulations for $k \leq 3$, k-angulations, k-regular partitions, k-regular straight line graphs for $k \leq 2$, spanning trees. Then there exists a perfect matching between the edge sets of P and Q such that matched edges share a vertex. The same is true for the face sets of P and Q, except for the last two classes and for rank-3 pseudo-triangulations.*

Let us demonstrate that an edge incidence matching need not exist for pseudo-triangulations of general (fixed) rank. See Figure 2. The two pseudo-triangulations we use are the one shown there (call it PT_1) and the one we obtain when reflecting PT_1 along the bold vertical edge (call this structure PT_2). Note that PT_1 and PT_2 live on the same point set. Let Δ denote the shaded triangle. Consider the restrictions of PT_1 and PT_2, respectively, to Δ, and let E_1 and E_2 be their respective edge sets. The 15 edges of E_1 can only be matched to the 11 edges of E_2 or to the 3 additional edges of PT_2 that are incident to the vertices of Δ. Thus no perfect matching is possible.

Note that Figure 2 serves as an example, that requiring $c_3 \geq -1$ instead of $c_3 \geq 0$ in Theorem 2 is not strong enough to ensure an incidence matching.

For triangulations, vertex incidence of matched triangles *plus* overlap can be satisfied simultaneously [2]. While the overlap condition has to be dropped for

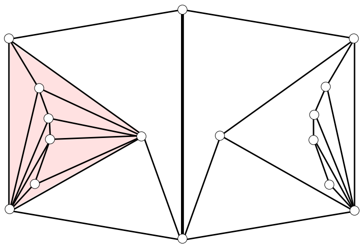

Figure 2: No edge matching exists for this rank-4 pseudo-triangulation and its reflection

general pseudo-triangulations, see Figure 1, the incidence condition for pseudo-triangles can be retained for rank $k \leq 2$, see Theorem 4. In particular, pointed pseudo-triangulations admit such a face matching.

5 Decomposition into spanning trees

Several authors considered the question of whether a given graph is decomposable into disjoint spanning trees; see e.g. [7] and references therein. Using a basic theorem by Nash-Williams [11] and Tutte [15], the following can be proved for polygonal partitions.

Theorem 5 *Let P be a polygonal partition on S with $k(n(S)-1)$ edges. The edge set of P can be decomposed into k spanning trees if and only if P is edge-Laman with characteristic $(k, 0, k)$.*

From Observation 1 we get the following property.

Corollary 6 *Every rank-1 pseudo-triangulation of S can be decomposed into two spanning trees.*

It is well known that, in case $conv(S)$ is a triangle, every triangulation of S is decomposable into three trees which are edge-disjoint apart from the three edges of $conv(S)$; see, e.g., [9, 13]. We obtain the following generalizations.

Corollary 7 *Every triangulation of S can be decomposed into 3 spanning trees if the $h(S)$ edges of $conv(S)$ are duplicated. Moreover, every quadrangulation of S can be decomposed into 2 spanning trees if every other edge of $conv(S)$ is duplicated.*

The existence of *some* edges in a triangulation (or quadrangulation) whose duplication leads to a decomposition into spanning trees also can be proved using a result in [7]. Duplication of *arbitrary* edges does not suffice, as can be shown by simple examples.

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