The Projection Median of a Set of Points in $\mathbb{R}^2$

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Abstract

We define the projection median of a non-empty and finite multiset of points in $\mathbb{R}^2$. We show the projection median provides a better approximation of the Euclidean (ℓ_2) median than do the rectilinear (ℓ_1) median or the centre of mass, both in terms of approximation factor and stability.

1 Introduction: Euclidean Median

Definition 1 *Given an arbitrary non-empty finite multiset P in $\mathbb{R}^2$, a* **Euclidean median** *of P is a point in $\mathbb{R}^2$, $M(P)$, that minimizes*

$$\sum_{p \in P} ||M(P) - p||. \tag{1}$$

If the points of P are not collinear, then the Euclidean median is unique [17]. Furthermore, M is invariant under similarity transformations.

The Euclidean median problem on three points in the plane was first posed by Fermat and solved by Torricelli early in the 17th century [16]. In $\mathbb{R}$, the Euclidean median is easily found in $\Theta(n)$ time, where $n = |P|$, by a linear-time selection algorithm. In two or more dimensions, the location of the Euclidean median cannot be solved exactly when $|P| \geq 5$ [3]. No polynomial-time algorithm is known, nor has the problem been shown to be NP-hard [12]. The most common approximation algorithm is Weiszfeld's algorithm [20], an iterative procedure that converges to the Euclidean median. Bose et al. [6] and Indyk [14] both give linear-time randomized algorithms for ϵ-approximations of the Euclidean median. Bose et al. [6] also give an $O(n \log n)$-time deterministic ϵ-approximation algorithm.

The Euclidean median has been repeatedly rediscovered under a variety of names. The most common of these is *Weber point.* Other names include Torricelli point, Fermat point, first Fermat point, isogonic centre, first isogonic centre, ℓ_2 median, 1-median, spatial median, Steiner point (amongst other definitions for a Steiner point, this one derives from the Steiner tree problem), the point of equilibrium in a Varignon frame, Kimberling triangle centre $X(13)$ [15], or any combination of Fermat-Steiner-Torricelli-Weber point. An overview of the history and solutions to the Euclidean median problem can be found in [9, 17, 21].

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2 Approximation Metrics

Point coordinates are commonly represented by discretization of real positions to nearby grid coordinates. That is, each point is approximated by the nearest grid point. Given a multiset of points P in $\mathbb{R}^2$, small perturbations at only a few points of P can result in a relatively large change (error) in the position of the Euclidean median of P. For example, let $P = \{(0,0),(0,0),(1,0),(1,\epsilon)\}$ and let $P' = \{(0,0),(0,\epsilon),(1,0),(1,0)\}$. For any $\epsilon > 0$, $M(P) = (0,0)$ and $M(P') = (1,0)$. In this sense, the Euclidean median is *unstable*.

Given this instability, the Euclidean median may be unfit for certain applications. A function that approximates the Euclidean median while maintaining some degree of stability may be better suited. We refer to such a function as a **median function**.

We formalize the notion of stability by defining κ-stability for a median function Υ as a bound on its maximum volatility. This requires preliminary definitions for an ϵ-perturbation and a continuous function.

Definition 2 *Given $\epsilon > 0$, function $f : P \to \mathbb{R}^2$ is an* **ϵ-perturbation** *on P if for all $p \in P$, $||p - f(p)|| \leq \epsilon$.*

Let F_ϵ^P denote the set of all ϵ-perturbations on P. A prerequisite for stability is continuity. Specifically, if the stability of median function Υ is bounded, then Υ must be continuous.

Definition 3 *A median function Υ is* **continuous** *if for all P in $\mathbb{R}^2$ and all $\delta > 0$ there exists an $\epsilon > 0$ such that for all $f \in F_\epsilon^P$,*

$$||\Upsilon(P) - \Upsilon(f(P))|| < \delta. \tag{2}$$

Definition 4 *Median function Υ is* **κ-stable** *if*

$$\forall \epsilon > 0,\ \forall f \in F_\epsilon^P,\ \kappa||\Upsilon(P) - \Upsilon(f(P))|| \leq \epsilon, \tag{3}$$

for all non-empty finite multisets P in $\mathbb{R}^2$.

The Euclidean median is not continuous. Consequently, it is not κ-stable for any $\kappa > 0$.

Similarly, we formalize the notion of approximation factor by defining λ-approximation for a median function Υ as a bound on its worst-case relative approximation of Eq. (1).

Definition 5 *Median function Υ is a* **λ-approximation** *of the Euclidean median, M, if*

$$\sum_{p \in P} ||p - \Upsilon(P)|| \leq \lambda \sum_{q \in P} ||q - M(P)||, \tag{4}$$

for all non-empty finite multisets P in $\mathbb{R}^2$.

3 Centre of Mass

Definition 6 *Given an arbitrary non-empty finite multiset P in $\mathbb{R}^2$, the* **centre of mass** *of P is the function whose value, $C(P)$, is the (unique) point in $\mathbb{R}^2$ given by*

$$C(P) = \frac{1}{|P|} \sum_{p \in P} p. \tag{5}$$

Function C is invariant under affine transformations. The centre of mass is easily found in $\Theta(n)$ time.

The centre of mass is also known as geometric centroid, centroid, mean, 1-mean, centre of gravity, and Kimberling triangle centre $X(2)$ [15]. The centre of mass is the point that minimizes the sum of the squares of distances to the points of P [19].

We now derive tight bounds on the approximation factor of the centre of mass. Lems. 1 and 2 and Thm. 3 refer to the following definitions for P, a, m, c, and x. Let P denote a finite multiset in $\mathbb{R}^2$ such that $a \neq M(P)$ for some $a \in P$. Let $a' = M(P)$, let $P' = (P - \{a\}) \cup \{a'\}$, and let $x = ||a - a'||$. Let $m = \sum_{p \in P} ||p - M(P)||$ and let $c = \sum_{p \in P} ||p - C(P)||$. Let m' and c' denote the corresponding values for P'.

Lemma 1 *Point $M(P)$ is a Euclidean median of P'.*

Proof. Assume false. That is, $M(P)$ is not a Euclidean median of P'. Thus,

$$\sum_{p \in P'} ||p - M(P')|| < \sum_{p \in P'} ||p - M(P)||.$$

Therefore,

$$\begin{aligned} \sum_{p \in P} ||p - M(P')|| &= \sum_{p \in P'} ||p - M(P')|| + x \\ &< \sum_{p \in P'} ||p - M(P)|| + x \\ &= \sum_{p \in P} ||p - M(P)||. \end{aligned}$$

Thus, $M(P)$ did not minimize $\sum_{p \in P} ||p - M(P)||$, deriving a contradiction. Therefore $M(P') = M(P)$. □

Lemma 2 *The ratio c'/m' is bounded by*

$$\frac{c - (n-1)\frac{x}{n} - (x - \frac{x}{n})}{m - x} \leq \frac{c'}{m'}. \tag{6}$$

Proof. Sum c' can be bounded from below by the maximum difference in $||p - C(P)||$ for each point $p \in P$. For point a this change is at most $\pm(x - x/n)$. For the remaining $n - 1$ points it is at most $\pm x/n$. Thus,

$$c - (n-1)\frac{x}{n} - \left(x - \frac{x}{n}\right) \leq c'.$$

By Lem. 1, $m' = m - x$ and Eq. (6) follows. □

Theorem 3 *The centre of mass provides a $(2 - 2/n)$-approximation of the Euclidean median.*

Proof. Assume P is a multiset that maximizes the approximation factor of C such that

$$\begin{aligned} c &> m\left(2 - \frac{2}{n}\right), \\ \Rightarrow cx - cm &> 2mx - 2m\frac{x}{n} - cm, \\ \Rightarrow \frac{c - (n-1)\frac{x}{n} - (x - \frac{x}{n})}{m - x} &> \frac{c}{m}, \\ \Rightarrow \frac{c'}{m'} &> \frac{c}{m}, \text{ by Lem. 2.} \end{aligned}$$

This contradicts our assumption that P maximizes the approximation factor of C. Therefore, $c \leq (2 - 2/n)m$. □

The bound is realized by $n - 1$ points located at the origin and a single point located at $(1, 0)$.

As shown by Bespamyatnikh et al. [4], any function defined by a convex combination of a set of mobile points moves with maximum relative velocity at most one. Since the centre of mass is a convex combination of the points of P, this result implies that the centre of mass is 1-stable. The bound is trivially tight, as demonstrated by any translation of the points of P.

4 Rectilinear Median

The rectilinear median is defined analogously to the Euclidean median with respect to the ℓ_1 norm instead of the ℓ_2 norm.

Definition 7 *Given an arbitrary non-empty finite multiset P in $\mathbb{R}^2$, a* **rectilinear median** *of P is a point in $\mathbb{R}^2$, $R(P)$, that minimizes*

$$\sum_{p \in P} ||R(P) - p||_1, \tag{7}$$

where $|| \cdot ||_1$ denotes the ℓ_1 norm.

Function R is invariant under translation and uniform scaling, but not under rotation or reflection.

The rectilinear median is found in $\Theta(n)$ time by solving two independent one-dimensional median problems on the x- and y-coordinates of the points of P.

Bespamyatnikh et al. [4], show that the relative velocity of the rectilinear median of a set of mobile points in $\mathbb{R}^2$ is at most $\sqrt{2}$. Furthermore, this bound is tight. It is straightforward to show that maximum relative velocity is inversely related to stability, implying that R is $(1/\sqrt{2})$-stable.

Bespamyatnikh et al. [4] also show that the rectilinear median provides a $\sqrt{2}$-approximation of the Euclidean median. We show this bound is tight even for small point sets in the following example. Let $2k$ points lie at $(1,0)$, let $k+1$ points lie at $(0,1)$, and let $k+1$ points lie at $(0,-1)$. When $k \geq 3$, the Euclidean median of P lies at $(1,0)$ and the (unique) rectilinear median of P lies at $(0,0)$. $\sum_{p \in P} ||p - R(P)|| = \sqrt{2} \sum_{q \in P} ||q - M(P)||$.

5 Projection Median

The definition of the Euclidean median is the natural generalization of the one-dimensional median to higher dimensions. Eq. (1), however, suggests other possible generalizations.

One possibility is to project points onto a line through the origin, to find the one-dimensional median of the projection, and to integrate these one-dimensional medians for all lines through the origin.

Let l_θ denote the line through the origin parallel to the unit vector $u_\theta = (\cos\theta, \sin\theta)$. Expressed in slope-intercept form, l_θ is the line $y = \tan\theta\ x$. Given a multiset of points P in $\mathbb{R}^2$ and an angle $\theta \in [0, \pi)$, let P_θ denote the multiset defined by the projection of P onto line l_θ. See Fig. 1A. That is,

$$P_\theta = \{u_\theta \langle p, u_\theta \rangle \mid p \in P\}. \tag{8}$$

Definition 8 *The* **projection median** *of a non-empty finite multiset P in $\mathbb{R}^2$ is*

$$\Pi(P) = \frac{2}{\pi} \int_0^\pi \text{med}(P_\theta)\ d\theta, \tag{9}$$

where $\text{med}(P_\theta)$ is the median of the projection of P onto the line $y = \tan\theta\ x$.

If $|P|$ is even, then P_θ may not have a unique median. In this case, let $\text{med}(P_\theta)$ denote the midpoint of the region of points on l_θ that define medians of P_θ. It is straightforward to show that Π is invariant under similarity transformations.

The formulation of the projection median displays some resemblance to the Steiner centre, which can be expressed similarly to Eq. 9 in $\mathbb{R}^2$ by replacing $\text{med}(P_\theta)$ with $\text{mid}(P_\theta)$, the midpoint of P_θ [10].

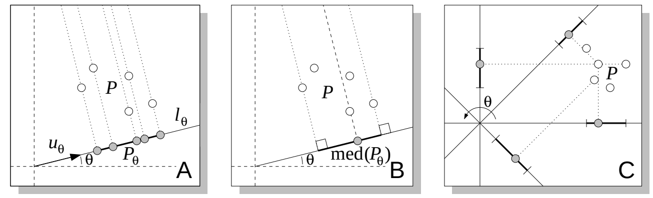

Figure 1: defining the projection median

Although this paper examines median functions defined over finite multisets, these can also be defined over bounded regions in $\mathbb{R}^d$ with an associated density function. In this case, the sums in Defs. 1 and 6–8 are replaced by integrals. This family of problems is referred to as *continuous facility location.* Fekete et al. [11] examine the continuous rectilinear median.

The projection median can be found using techniques similar to those developed by Bespamyatnikh et al. [5]. In brief, as θ varies from 0 to π, the point(s) in P that induce $\text{med}(P_\theta)$ are identified by maintaining a line (perpendicular to l_θ) that partitions P into two sets of at most $\lfloor n/2 \rfloor$ points each. The convex hull of each partition is maintained as the line rotates, requiring $O(\log^2 n)$ time per update [18]. Since the dual problem to maintaining these partitions corresponds to an $n/2$-level, we require at most $O(n^{4/3})$ such updates [8]. Between updates, the contribution to $\Pi(P)$ of the point(s) that induce $\text{med}(P_\theta)$ is calculated in $O(1)$ time. Together, these give an $O(n^{4/3} \log^2 n)$-time algorithm. This can be improved to $O(n^{4/3} \log^{1+\epsilon} n)$ amortized time using the dynamic convex hull data structure of Chan [7]. Providing details of this algorithm is not the goal of this paper; rather, we focus on the properties of approximation factor and stability.

Theorem 4 *The projection median provides a $(4/\pi)$-approximation of the Euclidean median.*

Proof. Let d_ϕ denote the ℓ_1 norm relative to a rotation by ϕ of the reference axis. That is, $d_\phi(x) = ||f_\phi(x)||_1$, where f_ϕ is a clockwise rotation about the origin by ϕ. Since $||x||_1 \geq ||x||$ for any x, similarly, $d_\phi(x) \geq ||x||$. Let $R_\phi = f_\phi^{-1}(R(f_\phi(P)))$ denote the rectilinear median with respect to norm d_ϕ. Observe that $R_\phi(P) = \text{med}(P_\phi) + \text{med}(P_{\phi+\pi/2})$. Consequently,

$$\Pi(P) = \frac{2}{\pi} \int_0^\pi \text{med}(P_\theta)\ d\theta = \frac{2}{\pi} \int_0^{\pi/2} R_\theta\ d\theta.$$

We bound the approximation factor of Π:

$$\frac{\sum_{p \in P} ||\Pi(P) - p||}{\sum_{q \in P} ||M(P) - q||} = \frac{\sum_{p \in P} \left|\left| \frac{2}{\pi} \int_0^{\pi/2} R_\theta(P)\ d\theta - p \right|\right|}{\sum_{q \in P} ||M(P) - q||}$$

$$\leq \frac{2}{\pi} \int_0^{\pi/2} \frac{\sum_{p \in P} ||R_\theta(P) - p||}{\sum_{q \in P} ||M(P) - q||}\ d\theta$$

$$\leq \frac{2}{\pi}\int_0^{\pi/2} \frac{\sum_{p\in P} d_\theta(R_\theta(P)-p)}{\sum_{q\in P}||M(P)-q||}\,d\theta$$
$$\leq \frac{2}{\pi}\int_0^{\pi/2} \frac{\sum_{p\in P} d_\theta(M(P)-p)}{\sum_{q\in P}||M(P)-q||}\,d\theta$$
$$= \frac{2}{\pi}\int_0^{\pi/2} \frac{\sum_{p\in P}(|\sin\theta|+|\cos\theta|)||M(P)-p||}{\sum_{q\in P}||M(P)-q||}\,d\theta$$
$$= \frac{2}{\pi}\int_0^{\pi/2} |\sin\theta|+|\cos\theta|\,d\theta$$
$$= \frac{4}{\pi}. \quad \square$$

Although we do not prove that this bound is tight, we give the following lower bound.

Theorem 5 *The projection median cannot guarantee an approximation factor less than* $\sqrt{4/\pi^2+1}$ *in the worst case.*

Proof. Let multiset P be defined by k points located at $(0,1)$, k points located at $(0,-1)$, and a single point located at $(x,0)$, for some $k\in\mathbb{N}$ and $x\in\mathbb{R}^+$. See Fig. 2.

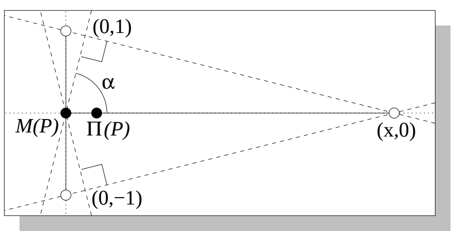

Figure 2: example realizing the bound in Thm. 5

By the symmetry of P, $M(P)$ must lie on the x-axis. Consequently, it is straightforward to show that $M(P) = (1/\sqrt{4k^2-1}, 0)$. By Eq. (9), the projection median of P is located at $\Pi(P) = (2x\arctan(1/x)/\pi, 0)$. The approximation factor λ is at least

$$\lambda \geq \lim_{\substack{x\to\infty\\k\to\infty}} \frac{\sum_{p\in P}||\Pi(P)-p||}{\sum_{q\in P}||M(P)-q||}$$
$$= \lim_{\substack{x\to\infty\\k\to\infty}} \frac{2k\sqrt{\frac{4x^2}{\pi^2}\arctan^2\left(\frac{1}{x}\right)+1}+x-\frac{2x}{\pi}\arctan\left(\frac{1}{x}\right)}{2k\sqrt{\frac{1}{4k^2-1}+1}+x-\frac{1}{\sqrt{4k^2-1}}}$$
$$= \sqrt{\frac{4}{\pi^2}+1}. \quad \square$$

Theorem 6 *The projection median is* $(\pi/4)$*-stable.*

Proof. Choose any non-empty and finite P in $\mathbb{R}^2$. Let $f: P\to\mathbb{R}^2$ be any ϵ-perturbation of P. Let multiset $Q = f(P)$. Since Π is invariant under rotation and translation, without loss of generality assume $\Pi(P)$ and $\Pi(Q)$ lie on the x-axis. The one-dimensional median is 1-stable. Consequently, for any ϵ-perturbation of P, f,

$$||\mathrm{med}(P_\theta)-\mathrm{med}(Q_\theta)|| \leq \max_{p\in P}||p-f(p)||.$$

Thus, for any θ,

$$\begin{aligned}|\mathrm{med}(P_\theta)_x-\mathrm{med}(Q_\theta)_x| &= |\cos\theta|\cdot||\mathrm{med}(P_\theta)-\mathrm{med}(Q_\theta)||\\ &\leq |\cos\theta|\cdot\max_{p\in P}||p-f(p)||\\ &\leq |\cos\theta|\cdot\epsilon,\end{aligned}$$

where a_x denotes the x-coordinate of a. We bound the stability of Π from below by

$$\begin{aligned}&||\Pi(P)-\Pi(f(P))||\\ =&|\Pi(P)_x-\Pi(Q)_x|\\ =&\left|\frac{2}{\pi}\int_0^\pi \mathrm{med}(P_\theta)_x\,d\theta - \frac{2}{\pi}\int_0^\pi \mathrm{med}(Q_\theta)_x\,d\theta\right|\\ \leq&\frac{2}{\pi}\int_0^\pi |\mathrm{med}(P_\theta)_x-\mathrm{med}(Q_\theta)_x|\,d\theta\\ \leq&\frac{2}{\pi}\int_0^\pi |\cos\theta|\cdot\epsilon\,d\theta\\ =&\frac{4\epsilon}{\pi}.\end{aligned}$$

Therefore, for all non-empty finite multisets P in $\mathbb{R}^2$,

$$\forall\epsilon>0,\ \forall f\in F_\epsilon^P,\ \frac{\pi}{4}||\Pi(P)-\Pi(f(P))||\leq\epsilon. \quad \square \quad (10)$$

The bound in Eq. (10) is shown to be tight by the following example. Let P be an even number of points uniformly distributed on the unit circle centred at the origin. Choose any $\epsilon\in(0,1)$ and define an ϵ-perturbation such that points above the x-axis move right (clockwise) in a direction tangent to the circle while points below the x-axis move right (counter-clockwise) in the opposite direction. Every point p in P has a corresponding point in P, $q=-p$, opposite the origin from p. The midpoint of each such pair of points p and q defines $\mathrm{med}(P_\theta)$ for some P_θ (corresponding to the projection onto the line perpendicular to $p-q$). The resulting change in the position of $\mathrm{med}(P_\theta)$ is identical to the change at p and q. The resulting stability corresponds exactly to that derived in equation Eq. (10).

6 Evaluation

As shown in Sec. 2, the Euclidean median, M, is highly unstable. Guaranteeing any degree of stability in a median function implies an increase in the sum of the distances in Eq. (1). The ratio of the sums of the distances defines the approximation factor. In this paper we introduce the projection median, Π, as a stable approximation of the Euclidean median. We now compare the stability and approximation factor of Π against those of two common median functions: the rectlinear median, R, and the centre of mass, C. See Tab. 1.

Observe that Π is more stable and guarantees a better approximation factor than R. Similarly, Π guarantees

median function	notation	approximation	stability
Euclidean median	M	1	0
rectilinear median	R	$\sqrt{2} \approx 1.41$	$1/\sqrt{2} \approx 0.71$
centre of mass	C	2	1
projection median	Π	$[\sqrt{4/\pi^2+1}, 4/\pi]$	$\pi/4$
		$\approx [1.18, 1.27]$	≈ 0.79

Table 1: comparing median functions in $\mathbb{R}^2$

a better approximation than C, but one that is not as stable.

Finally, Def. 8 has a natural generalization to $\mathbb{R}^d$, suggesting that the properties that make the projection median a good median function might not be limited to $\mathbb{R}^2$, but may extend to three or higher dimensions.

7 Applications to Mobile Facility Location

The projection median's benefits extend beyond its definition as the median of a set of static points. Recently, several questions of facility location have been posed within the setting of mobile facility location (e.g., [1, 2, 4, 13]). Given a set of mobile clients moving continuously and with bounded velocity in $\mathbb{R}^2$, the fitness of a mobile facility is determined both by its approximation factor and also by its maximum velocity and continuity of its motion. The stability of a median function is inversely related to the maximum velocity of a mobile facility, providing further motivation for the need of stability in a median function. Thus, the projection median defines the position of a mobile facility that approximates the mobile Euclidean median with a factor of $4/\pi$ while maintaining a maximum velocity of at most $4/\pi$ relative to the velocity of the clients.

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