Large k-D Simplices in the d-Dimensional Unit Cube

Gill Barequet* Jonathan Naor†

Abstract

In this paper we show lower and upper bounds for a generalization of Heilbronn's triangle problem to d dimensions. Namely, we show that there exists a set S_1 (resp., S_2) of n points in the d-dimensional unit cube such that the minimum-area triangle (embedded in d dimensions) defined by some three points of S_1 (resp., S_2) has an area of $\Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$ (resp., $O(d/n^{2/d})$). We then generalize the applied methods and show that there exists a set S_3 (resp., S_4) of n points in the d-dimensional unit cube such that the minimum-volume k-dimensional simplex (embedded in d dimensions, for $2 \leq k \leq d$) defined by some $k+1$ points of S_3 (resp., S_4) has volume $\Omega(f(k,d)/n^{k/(d-k+1)})$, where $f(k,d)$ is independent of n (resp., $O(k^{k/d}d^{k/2}/(k!\,n^{k/d}))$).

Keywords: Heilbronn's triangle problem, probabilistic method.

1 Introduction

Heilbronn's famous triangle problem, posed in the 1950s, posits the following:

Problem 1 *Given n points in the d-dimensional unit cube, what is $\mathcal{H}_{k,d}(n)$, the maximum possible volume of the* smallest *k-dimensional simplex defined by some $k+1$ (for integral $2 \leq k \leq d$) of these points?*

Heilbronn was interested in the area of triangles defined by points located in the unit square, that is, in the special case when $k = d = 2$. Erdős [9, appendix] showed that $\mathcal{H}_{2,2}(n) = \Omega(1/n^2)$ by a simple example (points on the moment curve). Thirty years later, Komlós, Pintz, and Szemerédi [5] showed by a rather involved probabilistic construction that $\mathcal{H}_{2,2}(n) = \Omega(\log n/n^2)$. A simpler construction (which we followed in [2] and also in the present paper) by Alon and Spencer [1] proves a weaker lower bound of $\Omega(1/n^2)$. It is trivial to show that $\mathcal{H}_{2,2}(n) = O(1/n)$: any triangulation of any point set (in general position) in the unit square admits $\Theta(n)$ triangles. Currently, the best known upper bound for the triangle problem, $\mathcal{H}_{2,2}(n) = O(1/n^{1.142\cdots})$, is due to Komlós, Pintz, and Szemerédi [4]. A comprehensive survey of the history of this problem (excluding the results of Komlós et al.) is given by Roth in [10].

In [2] we investigated $\mathcal{H}_{d,d}(n)$. Specifically, we showed that $\mathcal{H}_{d,d}(n) = \Omega(1/n^d)$ for a fixed value of d. This lower bound was achieved by both a specific example (points on the d-dimensional moment curve) and a probabilistic argument. (In fact, the moment-curve example shows that $\mathcal{H}_{d,d}(n) = \Omega(1/(d!\,n^d))$.) Lefmann [6] slightly improved this bound, showing, by using uncrowded hypergraphs, that $\mathcal{H}_{d,d}(n) = \Omega(\log n/n^d)$ (again, for a fixed value of d). Bertram-Kretzberg, Hofmeister, and Lefmann [3] showed that a specific point set that realizes the lower bound in two dimensions can be found in time polynomial in n. Lefmann and Schmitt [8] proved a similar result for three dimensions.

In the current paper we provide lower and upper bounds on $\mathcal{H}_{k,d}(n)$. Namely, we attempt to maximize the volume of the minimum-volume k-dimensional simplex in a d-dimensional unit cube.[1] As mentioned above, in the plane there is a large gap between the trivial lower and upper bounds for $\mathcal{H}_{2,2}(n)$, namely, $\Omega(1/n^2)$ and $O(1/n)$. In this paper we show by two different methods that $\mathcal{H}_{2,d}(n)$ is $\Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$ and $O(d/n^{2/d})$ ($\Omega(1/n^{2/(d-1)})$ and $O(1/n^{2/d})$ for a fixed value of d). We then generalize these methods and show that $\mathcal{H}_{k,d}(n)$ (for $2 \leq k \leq d$) is $\Omega(f(k,d)/n^{k/(d-k+1)})$ (where $f(k,d)$ is independent of n) and $O(k^{k/d}d^{k/2}/(k!\,n^{k/d}))$ ($\Omega(1/n^{k/(d-k+1)})$ and $O(1/n^{k/d})$ for fixed values of k and d).

The rest of this paper is organized as follows. In Sections 2 and 3 we prove the lower and upper bounds, respectively, on $\mathcal{H}_{k,d}(n)$. We end in Section 4 with some concluding remarks.

2 The Lower Bound

2.1 A Probabilistic Lemma

We first prove a lemma, which is a generalization of a probabilistic argument of Alon and Spencer [1, p. 30]. A slightly weaker version of this lemma was proven in [2],

*Dept. of Computer Science, The Technion—Israel Institute of Technology, `[barequet,jnaor]@cs.technion.ac.il`

[1] To avoid confusion, we call the generalization of Heilbronn's problem to higher dimensions the "simplex problem," and reserve the term "triangle problem" for instances of the problem in which the sought simplex is two-dimensional.

so for completeness we provide the modified (and short) proof here.

Lemma 1 *Let $H(P_1, P_2, \dots, P_m)$ be a mapping from m-tuples of points $P_1, P_2, \dots, P_m$ in some domain $\mathcal{D}$ to $\Re^+ \cup \{0\}$. If there exist constants $c_1 > 0$, c_2 such that $\text{Prob}[H(P_1, P_2, \dots, P_m) \le \varepsilon] \le c_1 \varepsilon^{c_2}$, where $P_1, P_2, \dots, P_m$ are chosen randomly, uniformly, and independently in $\mathcal{D}$, then there exists a set S of n points in $\mathcal{D}$ and a constant $c_3 = (\frac{m!}{2^m c_1})^{1/c_2} > 0$ such that $\min_{P_{i_1}, P_{i_2}, \dots, P_{i_m} \in S} H(P_{i_1}, P_{i_2}, \dots, P_{i_m}) > c_3 n^{-\frac{m-1}{c_2}}$.*

Proof. Let $P_1, P_2, \dots, P_{2n}$ be a set of $2n$ points selected randomly, uniformly, and independently in $\mathcal{D}$. Fix $c_3 = (\frac{m!}{2^m c_1})^{1/c_2}$. Let the random variable X count the number of m-tuples $P_{i_1}, P_{i_2}, \dots, P_{i_m}$ for which $H(P_{i_1}, P_{i_2}, \dots, P_{i_m}) \le c_3 n^{-\frac{m-1}{c_2}}$. Then, $E[X] \le \binom{2n}{m} c_1 (c_3 n^{-\frac{m-1}{c_2}})^{c_2} < \frac{(2n)^m}{m!} \cdot \frac{m!}{2^m n^{m-1}} = n$. Therefore, there exists a specific set of $2n$ points with fewer than n m-tuples $P_{i_1}, P_{i_2}, \dots, P_{i_m}$ for which $H(P_{i_1}, P_{i_2}, \dots, P_{i_m}) \le c_3 n^{-\frac{m-1}{c_2}}$. Remove one point of the set from each such m-tuple. (The same point may be deleted more than once but this only helps.) This leaves at least n points and now all m-tuples $P_{i_1}, P_{i_2}, \dots, P_{i_m}$ satisfy $H(P_{i_1}, P_{i_2}, \dots, P_{i_m}) > c_3 n^{-\frac{m-1}{c_2}}$. □

Alon and Spencer [1, p. 30] proved the special case of Lemma 1 in which $c_2 = 1$ and $m = 3$, and used it for showing that $\mathcal{H}_{2,2}(n) = \Omega(1/n^2)$. The generalization of the lemma in [2] unnecessarily linked m, the number of arguments of the function H, and d, the dimension of the space in which the points are located, by assuming $m = d + 1$.

2.2 Triangles in d Dimensions

We introduce our technique by first using it for planar simplices, that is, triangles.

Theorem 2 $\mathcal{H}_{2,d}(n) = \Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$.

Proof. Let $A(P_{i_0}, P_{i_1}, P_{i_2})$ be the area of the triangle defined by P_{i_0}, P_{i_1}, and P_{i_2}. We first upper bound $\text{Prob}[A(P_{i_0}, P_{i_1}, P_{i_2}) \le \varepsilon]$. Let x be the distance from P_{i_0} to P_{i_1}. Then,

$$\begin{aligned}\text{Prob}[b \le x \le b + \mathrm{d}b] &\le \mathrm{d}(\pi^{d/2} b^d / \Gamma(d/2+1)) \\ &= (\pi^{d/2} d b^{d-1} / \Gamma(d/2+1))\, \mathrm{d}b,\end{aligned}$$

the difference[2] between the volumes of the corresponding balls in $\Re^d$.[3] Given P_{i_0} and P_{i_1} at distance b, the altitude h from P_{i_2} to the line defined by P_{i_0} and P_{i_1} satisfies $bh/2 \le \varepsilon$, i.e., $h \le 2\varepsilon/b$. Thus, P_{i_2} must lie within a d-dimensional cylinder whose height is at most $\sqrt{d}$ and whose cross-section is a $(d-1)$-dimensional ball whose volume is $\pi^{(d-1)/2}(2\varepsilon/b)^{d-1}/\Gamma((d+1)/2)$. This occurs with probability at most $\pi^{(d-1)/2}\sqrt{d}(2\varepsilon/b)^{d-1}/\Gamma((d+1)/2)$. Since $0 \le b \le \sqrt{d}$,

$$\begin{aligned}&\text{Prob}[A(P_{i_0}, P_{i_1}, P_{i_2}) \le \varepsilon] \\ &\le \int_0^{\sqrt{d}} \left(\frac{\pi^{d/2} d b^{d-1}}{\Gamma(d/2+1)} \right) \left(\frac{\pi^{(d-1)/2}\sqrt{d}(2\varepsilon/b)^{d-1}}{\Gamma((d+1)/2)} \right) \mathrm{d}b \\ &= \frac{\pi^{d-1/2} 2^{d-1} d^2 \varepsilon^{d-1}}{\Gamma(d/2+1)\Gamma((d+1)/2)}.\end{aligned}$$

Now apply Lemma 1 with $c_1 = \pi^{d-1/2} 2^{d-1} d^2 / (\Gamma(d/2+1)\Gamma((d+1)/2))$, $c_2 = d-1$, and $m = 3$, and conclude that there exists a set $S \subset [0,1]^d$ of n points for which $\min_{P_{i_0}, P_{i_1}, P_{i_2} \in S} A(P_{i_1}, P_{i_2}, P_{i_3}) > c_3 / n^{2/(d-1)}$, where

$$\begin{aligned}c_3 &= \left(\frac{3!\,\Gamma(d/2+1)\Gamma((d+1)/2)}{2^{d+2}\pi^{d-1/2} d^2} \right)^{\frac{1}{d-1}} \\ &= \Theta(d^{1-1/(2(d-1))})\end{aligned}$$

(by applying Stirling's asymptotic approximation $\Gamma(x+1) \sim \sqrt{2\pi x}(x/e)^x$). That is, $\mathcal{H}_{2,d}(n) = \Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$. □

2.3 k-Dimensional Simplices

We now use the technique of Section 2.2 to show our main result:

Theorem 3 $\mathcal{H}_{k,d}(n) = \Omega(f(k,d)/n^{k/(d-k+1)})$, *where $f(k,d)$ is a function of only k and d that is independent of n.*

Proof. Let $P_0, P_1, \dots, P_k$ be $k+1$ points in the d-dimensional unit cube, and let $V(P_0, P_1, \dots, P_k)$ denote the volume of the k-dimensional simplex defined by these points. Also denote by x_i (for $1 \le i \le k$) the distance from P_i to E_{i-1}, the $(i-1)$-dimensional flat defined by $P_0, P_1, \dots, P_{i-1}$.

Let us begin with upper bounding $\text{Prob}[V(P_0, P_1, \dots, P_k) < \varepsilon]$. First,

$$\begin{aligned}\text{Prob}[b_1 \le x_1 \le b_1 + \mathrm{d}b_1] &\le \mathrm{d}\left(\frac{\pi^{\frac{d}{2}} b_1^d}{\Gamma(\frac{d}{2}+1)}\right) \\ &= \frac{\pi^{\frac{d}{2}} d b_1^{d-1}}{\Gamma(\frac{d}{2}+1)} \mathrm{d}b_1,\end{aligned}$$

the difference between the volumes of the corresponding balls. Second,

$$\begin{aligned}\text{Prob}[b_2 \le x_2 \le b_2 + \mathrm{d}b_2] &\le \mathrm{d}\left(\sqrt{d}\frac{\pi^{\frac{d-1}{2}} b_2^{d-1}}{\Gamma(\frac{d-1}{2}+1)}\right) \\ &= \sqrt{d}\frac{\pi^{\frac{d-1}{2}}(d-1) b_2^{d-2}}{\Gamma(\frac{d-1}{2}+1)} \mathrm{d}b_2,\end{aligned}$$

[2] We use the non-italicized symbol ‘d’ to denote the differentiation operator, in order to avoid confusion with the italicized symbol ‘d’ that denotes the dimension.

[3] Recall that the volume of a d-dimensional ball, with radius r, is $\pi^{d/2} r^d / \Gamma(d/2+1)$, where $\Gamma(\cdot)$ is the continuous generalization of the factorial function, for which $\Gamma(x) = (x-1)\Gamma(x-1)$, $\Gamma(1/2) = \sqrt{\pi}$, and $\Gamma(1) = 1$. It is easy to see that $\text{vol}(B^{2k}) = \pi^k/k!$ and $\text{vol}(B^{2k+1}) = \pi^{k+1/2}/\Gamma(k+3/2)$ (where B^d denotes the unit d-dimensional ball), for an integral $k \ge 1$.

the difference between the volumes of the corresponding cylindrical shapes obtained by sweeping $(d-1)$-dimensional balls along a straight path whose length is at most that of the main diagonal of the unit cube. The general probability term is thus

$$\mathrm{Prob}[b_i \le x_i \le b_i + \mathrm{d}b_i] \le \mathrm{d}\Big(d^{\frac{i-1}{2}} \frac{\pi^{\frac{d-i+1}{2}} b_i^{d-i+1}}{\Gamma(\frac{d-i+1}{2}+1)}\Big)$$
$$= d^{\frac{i-1}{2}} \frac{\pi^{\frac{d-i+1}{2}} (d-i+1) b_i^{d-i}}{\Gamma(\frac{d-i+1}{2}+1)} \, \mathrm{d}b_i.$$

For the second-to-last point we have

$$\mathrm{Prob}[b_{k-1} \le x_{k-1} \le b_{k-1} + \mathrm{d}b_{k-1}]$$
$$\le \mathrm{d}\Big(d^{\frac{k-2}{2}} \frac{\pi^{\frac{d-k+2}{2}} b_{k-1}^{d-k+2}}{\Gamma(\frac{d-k+2}{2}+1)}\Big)$$
$$= d^{\frac{k-2}{2}} \frac{\pi^{\frac{d-k+2}{2}} (d-k+2) b_{k-1}^{d-k+1}}{\Gamma(\frac{d-k+2}{2}+1)} \, \mathrm{d}b_{k-1}.$$

For the last point we have the condition $\prod_{i=1}^{k} \frac{b_i}{k!} \le \varepsilon$, that is, $b_k \le \frac{k!\,\varepsilon}{\prod_{i=1}^{k-1} b_i}$. Therefore, P_k must lie in a shape whose volume is at most

$$d^{\frac{k-1}{2}} \frac{\pi^{\frac{d-k+1}{2}}}{\Gamma(\frac{d-k+1}{2}+1)} \left(\frac{k!\,\varepsilon}{\prod_{i=1}^{k-1} b_i}\right)^{d-k+1}$$

(the product of a $(k-1)$-dimensional slab, each of whose dimensions is at most $\sqrt{d}$, and a $(d-k+1)$-dimensional ball).

The probability of obtaining a k-dimensional simplex of volume at most ε is thus upper bounded by

$$\underbrace{\int_0^{\sqrt{d}}\int_0^{\sqrt{d}} \cdots \int_0^{\sqrt{d}}}_{k-1 \text{ integrations}} \Big(\pi^{\sum_{i=d-k+1}^{d} \frac{i}{2}} \cdot d^{\sum_{i=1}^{k-1} \frac{i}{2}} \cdot d! \tag{1}$$
$$\cdot\, b_1^{k-2} b_2^{k-3} \ldots b_{k-3}^2 b_{k-2}$$
$$\cdot\, (k!)^{d-k+1} \cdot \varepsilon^{d-k+1}\Big)$$
$$/\Big(\prod_{i=d-k+1}^{d} \Gamma\left(\tfrac{i}{2}+1\right) \cdot (d-k+1)!\Big)$$
$$\mathrm{d}b_1 \mathrm{d}b_2 \ldots \mathrm{d}b_{k-1}.$$

By Stirling's approximation,

$$\prod_{i=d-k+1}^{d} \Gamma\left(\tfrac{i}{2}+1\right) \sim \prod_{i=d-k+1}^{d} \left(\sqrt{2\pi \tfrac{i}{2}} \cdot \frac{(\frac{i}{2})^{\frac{i}{2}}}{e^{\frac{i}{2}}}\right) \tag{2}$$
$$= \frac{\pi^{\frac{k}{2}} \sqrt{\frac{d!}{(d-k)!} \prod_{i=d-k+1}^{d} i^i}}{(2e)^{k(2d-k-1)/4}}.$$

After substituting Eq. (2) in Eq. (1) and integrating $k-1$ times, we conclude that the probability of obtaining a k-dimensional simplex of volume at most ε is at most

$$\frac{\pi^{k(2d-k-1)/4} d^{k(k-1)/4} \sqrt{d!}(2e)^{k(2d-k+1)/4} (k!)^{d-k+1}}{\sqrt{\prod_{i=d-k+1}^{d} i^i}\,(d-k+1)\sqrt{(d-k)!}(k-1)!}$$
$$\cdot\, \varepsilon^{d-k+1}.$$

Finally, set

$$c_1 = \frac{\pi^{k(2d-k-1)/4} \cdot d^{k(k-1)/4} \cdot \sqrt{d!} \cdot (2e)^{k(2d-k+1)/4} \cdot (k!)^{d-k+1}}{\sqrt{\prod_{i=d-k+1}^{d} i^i} \cdot (d-k+1) \cdot \sqrt{(d-k)!} \cdot (k-1)!},$$

$c_2 = d-k+1$, and $m = k+1$, and apply Lemma 1. The lemma tells us that there exists a set of n points of which every subset of $k+1$ points defines a k-dimensional simplex whose volume is at least $c_3/n^{k/(d-k+1)}$, where $c_3 = \left(\frac{(k+1)!}{2^{k+1} c_1}\right)^{1/(d-k+1)}$. Let us finally give a lower bound on c_3. By substituting c_1 in the above term, we see that

$$c_3^{d-k+1} = \tag{3}$$
$$\frac{(k+1)!(k-1)!}{(k!)^{d-k+1}}$$
$$\cdot\, \frac{(d-k+1)\sqrt{\prod_{i=d-k+1}^{d} i^i}}{2^{k+1+k(2d-k+1)/4} e^{k(2d-k+1)/4} \pi^{k(2d-k-1)/4} d^{k(k-1)/4}}$$
$$\cdot\, \frac{1}{\sqrt{\prod_{i=d-k+1}^{d} i}}.$$

We write

$$\frac{(k+1)!(k-1)!}{(k!)^{d-k+1}} = \frac{k+1}{k} \cdot \frac{1}{(k!)^{d-k-1}} \ge \frac{1}{(k!)^{d-k-1}}$$

and

$$(k!)^{d-k-1} \sim$$
$$2^{(d-k-1)/2} \pi^{(d-k-1)/2} e^{-k(d-k-1)} k^{(2k+1)(d-k-1)/2},$$

and so,

$$\frac{(k+1)!(k-1)!}{(k!)^{d-k+1}}$$
$$\ge \frac{1}{2^{(d-k-1)/2} \pi^{(d-k-1)/2} e^{-k(d-k-1)} k^{(2k+1)(d-k-1)/2}}.$$

In addition,

$$\prod_{i=d-k+1}^{d} i^i \ge (d-k+1)^{k(2d-k+1)/2}$$

and

$$\sqrt{\prod_{i=d-k+1}^{d} i} \le d^{\frac{k}{2}}.$$

Substituting all these terms in Equation (3), we obtain that

$$c_3^{d-k+1} \ge$$
$$e^{k(2d-3k-5)/4} (d-k+1)^{k(2d-k+1)/4+1} /$$
$$\big(2^{(2kd-k^2+2d+3k+2)/4} \pi^{(2kd-k^2+2d-3k-2)/4}$$
$$k^{(2kd-2k^2+d-3k-1)/2} d^{k(k-1)/4}\big),$$

concluding that

$$c_3 \ge \frac{e^{\frac{k}{2} - \frac{k(k+7)}{4(d-k+1)}} (d-k+1)^{\frac{k}{2} + \frac{k^2-k+5}{4(d-k+1)}}}{2^{\frac{k+1}{2} + \frac{k(k+3)}{4(d-k+1)}} \pi^{\frac{k+1}{2} + \frac{(k+1)(k-4)}{4(d-k+1)}} k^{k+\frac{1}{2} - \frac{2k+1}{d-k+1}} d^{\frac{k(k-1)}{4}}}$$
$$= \left(\frac{e^{\frac{k}{2}} (d-k+1)^{\frac{k}{2}}}{(2\pi)^{\frac{k+1}{2}} k^{k+\frac{1}{2}} d^{\frac{k(k-1)}{4}}}\right)$$
$$\cdot \left(\frac{(d-k+1)^{\frac{k^2-k+5}{4}} k^{2k+1}}{2^{\frac{k(k+3)}{4}} e^{\frac{k(k+7)}{4}} \pi^{\frac{(k+1)(k-4)}{4}}}\right)^{\frac{1}{d-k+1}}.$$

This completes the proof. □

Note that substituting $k = 2$ in this bound yields $\mathcal{H}_{2,d}(n) = \Omega(d^{1/2+7/(4(d-1))}/n^{2/(d-1)})$, which is a bit weaker than the bound $\Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$ shown in the previous section.

3 The Upper Bound

Again, we first demonstrate the argument for $k = 2$:

Theorem 4 $\mathcal{H}_{2,d}(n) = O(d/n^{2/d})$.

Proof. The upper bound is set by a pigeonhole argument. Cover the d-dimensional unit cube with a regular grid whose step is $1/m^{1/d}$, where $m = (n-1)/2$. This divides the d-dimensional unit cube into exactly m small grid cubes. Put $n = 2m + 1$ points in the unit cube. There is at least one grid cube that contains at least three points. Since the length of the main diagonal of a grid cube is $\sqrt{d}/m^{1/d}$, the area of the triangle defined by these three points is $O(d/m^{2/d}) = O(2^{2/d}d/n^{2/d}) = O(d/n^{2/d})$. □

We use a similar argument for a general value of k:

Theorem 5 $\mathcal{H}_{k,d} = O(k^{k/d}d^{k/2}/(k!\,n^{k/d}))$.

Proof. Again, cover the d-dimensional unit cube with a regular grid whose step is $1/m^{1/d}$, where this time $m = (n-1)/k$. This divides the d-dimensional unit cube into exactly m small grid cubes. Put $n = km + 1$ points into the unit cube. There is at least one grid cube that contains at least $k + 1$ points. Since the length of the main diagonal of a grid cube is $\sqrt{d}/m^{1/d}$, the volume of the simplex defined by these $k + 1$ points is $O(d^{k/2}/(k!\,m^{k/d})) = O(k^{k/d}d^{k/2}/(k!\,n^{k/d}))$. □

4 Conclusion

In this paper we set lower and upper bounds on $\mathcal{H}_{k,d}(n)$, the maximum possible volume of the minimum-volume simplex defined by any k points that belong to a set of n points located in the d-dimensional unit cube. For $\mathcal{H}_{2,d}(n)$, the obtained bounds are $\Omega(d^{1-1/(2(d-1))}/n^{2/(d-1)})$ and $O(d/n^{2/d})$. (For a fixed value of d, these bounds are $\Omega(1/n^{2/(d-1)})$ and $O(1/n^{2/d})$, respectively.) For the general case, we obtain that $\mathcal{H}_{k,d}(n)$ (for $2 \leq k \leq d$) is $\Omega(f(k,d)/n^{k/(d-k+1)})$ (where $f(k,d)$ is a function that is independent of n) and $O(k^{k/d}d^{k/2}/(k!\,n^{k/d}))$. (For fixed values of k and d, these bounds are $\Omega(1/n^{k/(d-k+1)})$ and $O(1/n^{k/d})$, respectively.)

Recently, Lefmann [7] announced similar results for $k = 2$ by showing that $\mathcal{H}_{2,d}(n) = \Omega((\log n)^{1/(d-1)}/n^{2/(d-1)})$ and $\mathcal{H}_{2,d}(n) = O(1/n^{2/d})$. His lower bound is an improvement over our bound by a factor of $(\log n)^{1/(d-1)}$. The upper bound is identical to ours but is obtained by using a more difficult method. Both bounds do not specify the asymptotic dependence on d.

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