Flow Complex: General Structure and Algorithm*

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Abstract

The flow complex is a data structure, similar to the Delaunay triangulation, to organize a set of (weighted) points in $\mathbb{R}^d$. Its structure has been examined in detail in two and three dimensions but only little is known about its structure in general. Here we propose the first algorithm for computing the flow complex in any dimension which reflects its recursive structure. On the basis of the algorithm we give a generalized and simplified proof of the homotopy equivalence of alpha- and flow-shapes.

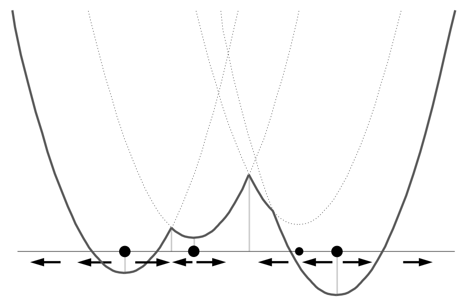

Figure 1: One-dimensional example of flow induced by a set of weighted points.

1 Introduction

The flow complex of a set of points has been successfully applied to surface reconstruction from a point cloud [5], to shape segmentation and matching [1], and to modeling properties of macromolecules in bio-geometry [4]. It is a cell complex based on the flow in the direction of the steepest ascent of the power distance function to a given set of weighted points. It is therefore closely related to the Voronoi diagram and the Delaunay triangulation, i.e., in the case of weighted points to the power diagram and the regular triangulation (see e.g. [8]).

So far the flow complex was only defined in two and three dimensions [4, 5]. The purpose of this paper is to give insight into the general structure of the complex independent of dimension. These insights lead to an algorithm for computing the flow complex of (weighted) points in any dimension, which reveals a recursive structure of the flow complex. We use this recursive structure to give a general proof of the homotopy equivalence of flow- and α-shapes [2].

2 Induced Flow and Flow Complex

Given a set of weighted points $P = \{(p, w_p) \in \mathbb{R}^d \times \mathbb{R}\}$. Let $\pi_p : \mathbb{R}^d \to \mathbb{R}, x \mapsto \|x - p\|^2 - w_p$ be the *(power) distance* to p. We define a vector field $v : \mathbb{R}^d \to \mathbb{R}^d$ as follows:

For any point $x \in \mathbb{R}^d$ let $A(x) \subset P$ be nearest neighbors of x in P. The point $d(x) = \text{argmin}_{y \in \text{conv}(A(x))} \|x - y\|^2$ is called the *driver* of x. The vector $v(x)$ is defined as $x - d(x)/\|x - d(x)\|$ if $d(x) \neq x$ and 0 otherwise. One can show that $v(x) \neq 0$ always points in the direction of steepest ascent of the *distance function* $h(x) = \min\{\pi_p(x) \,|\, p \in P\}$ at x, otherwise, i.e., if $v(x) = 0$, there is no direction of steepest ascent of h at x and we call x a *critical point* of h. Our notion of a critical point is in accordance with a well developed theory of critical points of distance functions, see [6].

Consider the one-dimensional example in Figure 1. The input points in P are shown as black dots, the weight w_p of each point is the signed distance from the point to the apex of the parabola below (positive weight) or above (negative weight) it. The graph of the distance function h is the lower envelope of the parabolas and the vector field v is indicated by arrows.

The *flow complex* is a decomposition of $\mathbb{R}^d$ into cells based on the flow along the vector field v. A point that follows this flow either reaches a point x with $v(x) = 0$, i.e., a critical point of h, from which it cannot escape, or it leaves any bounded region of $\mathbb{R}^d$ in finite time, i.e., the point flows to infinity. Thus the critical points of h are the *fixed points* of the flow.

The *stable manifold* of such a fixed point is the set of all points in $\mathbb{R}^d$ that flow into the point. The cells of the *flow complex* are the closures of the stable manifolds of the fixed points.

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3 Algorithm

Since the flow complex consists of the closures of the stable manifolds of the fixed points of the distance function, we have to compute all these closures in order to compute the flow complex. The algorithm that we are going to present makes use of the close relationship of the flow complex to the Delaunay and Voronoi diagram of P In the following the terms Delaunay and Voronoi diagram always include also the weighted versions.

Observation 1

(1) The critical points of the distance function h are the intersection points of Delaunay objects and their dual Voronoi objects.

(2) The driver of a point $x \in \mathbb{R}^d$ is the point closest to x in the Delaunay object dual to the lowest dimensional Voronoi object containing x.

(3) All points in the relative interior of a Voronoi object have the same driver.

(4) Let V, V' be Voronoi objects with $V' \subsetneq V$, let dr be the driver of V. If no line segment connecting a point of V' with dr intersects the relative interior of V then dr is also the driver of V'.

(5) The flow into a Voronoi object must come from Voronoi objects containing this object or through its boundary.

See [7] for proofs of these observations. The following algorithm builds on these observations and yields a description of the closure of a stable manifold as a polyhedral complex. For a convex set C let $ri(C)$ denote its relative interior. Figure 2 illustrates the algorithm by the example of a two–dimensional weighted point set.

INFLOW(CONVEX POLYTOPE P, VORONOI OBJECT V)
1 $inflow := \{P\}$
2 **for each** Voronoi obj. V' with $V \subsetneq V'$ **do**
3 $dr :=$ driver for the Voronoi object V'
4 $pyr := ri(conv(P, dr))$
5 **if** $pyr \cap V' \not\subseteq P$ **do**
6 $\mathcal{V} := \{\text{Vor. obj. } V'' \subseteq V' \mid pyr \cap ri(V'') \not\subseteq V\}$
7 **for each** $V'' \in \mathcal{V}$ **do**
8 $inflow := inflow \cup \text{INFLOW}(\overline{pyr} \cap V'', V'')$
9 **end for**
10 **end if**
11 **end for**
12 **return** $inflow$

The algorithm works as follows: It is called with two arguments, a convex polytope P and a Voronoi object V with the property that the relative interior of P is contained in the relative interior of V. We want to compute the closure of the inflow area of the relative interior of P of flow coming from Voronoi objects $V' \subseteq V$. We assume that the flow coming from V itself is already in P. Relative interiors are used to get a unique decomposition of the Voronoi diagram and the stable manifolds.

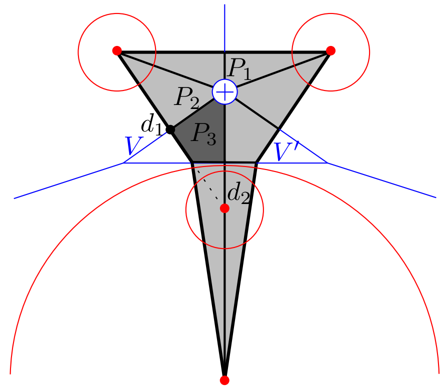

Figure 2: A decomposition of a stable manifold as computed by the algorithm with non-simplicial cells. The recursive construction of such a cell (dark grey): P_1 is a maximum of the distance function, $P_2 = conv(P_1, d_1) \subset V$, and $P_3 = conv(P_2, d_2) \cap V'$, where V is a Voronoi edge and V' is a Voronoi cell, whose drivers are d_1 and d_2, respectively.

Since the inflow has to contain P we add P to it in line 1 of the algorithm. To compute a stable manifold, we initially call the function INFLOW with the parameters c and V, where c is a critical point of h, i.e., a fixed point of the flow, and V is the lowest dimensional Voronoi object that contains c (Observation 1.1).

In the general case we have to take care of the inflow into the relative interior of P that comes through the boundary of V or through higher dimensional Voronoi objects that contain V in their boundary (Observation 1.5). Since the algorithm in line 2 only takes care of the higher dimensional Voronoi objects we need to guarantee that any flow coming through the boundary of V has been handled when INFLOW is called for P.

Note that in the special case of $P = \{c\}$ there cannot be any inflow from within the Voronoi object V (and thus from the boundary of V) since in this case c is the unique driver of the relative interior of V that repels all other points in this relative interior (Observation 1.3).

In the loop enclosed by lines 2 and 11 we take care of the inflow via all Voronoi objects V' that contain V in their boundary. The relative interior of any Voronoi object V' has a unique driver dr (Observation 1.3) that we determine in line 3 (using Observation 1.2). All points that flow via V' into the relative interior of P have to be contained in the intersection of V' with the relative interior pyr of the pyramid whose apex is dr and whose base is P. If V' does not contain its driver dr , i.e., if dr is not a critical point, then the whole pyramid cannot be contained in V' but is cut-off at the boundary of V'. This can result in a non–simplicial cell as in the example of Figure 2. In lines 5 to 10 we take care of the inflow coming from V' and its boundary. By definition V' is in $\mathcal{V}$ therefore the inflow into the cut-off pyramid coming from higher-dimensional Voronoi objects is computed by a recursive call of the algorithm in line 9. By construction there is no additional flow from the relative interior of V' (Observation 1.3).

We now handle flow into the cut–off pyramid coming from the boundary of V' by considering all possible cases. The polytope P is not driven into V'. If the driver dr of V' lies on the boundary of V' has to be a fixed point and is therefore not driven into V'. Any further point that is in the closure of *pyr* but outside *pyr* is driven past it by the common driver dr (Observation 1.4). Therefore, any flow coming from the boundary of V' must come from points in *pyr* which are taken care of in line 6. The recursion stops when there is no more inflow through higher dimensional Voronoi objects or through the boundary of a Voronoi object to consider.

4 Homotopy Equivalence

In the following we generalize and simplify the proof of the homotopy equivalence of α- and flow-shapes using the recursive structure of the flow-complex as it is inherently described in the above algorithm. Let us first review some definitions.

Union of balls and α-complex. Let P be a finite set of weighted points in $\mathbb{R}^d$. Let $B^\alpha(P)$ be the set of balls centered at the points in P with radius $\sqrt{\alpha + w_p}$, provided $p \in P_\alpha := \{p \in P \,|\, \alpha + w_p \geq 0\}$ The union of balls is the underlying space $|B^\alpha(P)| := \bigcup_{b \in B^\alpha(P)} b$ $= \{x \in \mathbb{R}^3 : \exists\, p \in P_\alpha$ such that $\pi_p(x) \leq \alpha\}$.

The α-complex $K^\alpha(P)$ of P is the dual complex of the Voronoi diagram of P restricted to the union of balls $|B^\alpha(P)|$. By construction the α-complex is a subcomplex of the Delaunay complex for every $\alpha \geq 0$. In fact the family of α-complexes is a *filtration* of the Delaunay complex. The underlying space of an α-complex is called α-shape.

The following theorem is due to Edelsbrunner [3].

Theorem 1 *For every $\alpha \geq 0$ the union of balls $|B^\alpha(P)|$ and the α-shape $|K^\alpha(P)|$ are homotopy equivalent.*

We also have a natural filtration of the flow complex. We denote the sub-complex of the flow complex that contains all stable manifolds of critical points at which the distance function h takes a value no more than $\alpha \geq 0$ as $F^\alpha(P)$. The underlying space $|F^\alpha(P)|$ is called flow-shape.

By Theorem 1 it is sufficient to prove the homotopy equivalence of $|F^\alpha(P)|$ and $|B^\alpha(P)|$. By definition the flow shapes do not change between the critical levels of the distance function. From the critical point theory of distance functions [6] we get that the homotopy of the union of balls does not change between critical levels:

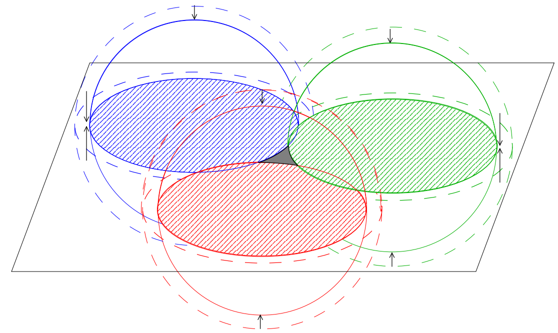

Figure 3: The dashed spheres can be retracted to the smaller spheres with the gray part glued in by recursively retracting them in directions of hyperplanes containing the centers.

Theorem 2 *If the interval $[\alpha, \alpha'] \subset [0, \infty)$ does not contain any critical level of h, i.e., there is no critical point $x \in \mathbb{R}^d$ of h with $h(x) \in [\alpha, \alpha']$, then $|B^\alpha(P)|$ is homeomorphic to $|B^{\alpha'}(P)|$, and $|B^\alpha(P)|$ is a deformation retract of $|B^{\alpha'}(P)|$.*

We therefore only need to consider the situation at critical levels. The following lemma describes how the homotopy of the union of balls changes at critical levels. It is illustrated in Figure 3. It generalizes an observation by Siersma [9] for the union of circles in two-dimensions.

Lemma 3 *Let α be a critical level of the distance function and x the only critical point of level α. Assume x is not a minimum. Let $\epsilon > 0$ be chosen such that the ϵ-neighborhood of α does not contain any critical level of h except α. Let D be the Delaunay object defining x (see Observation 1.1) and let $D_\epsilon := \text{closure}(D \setminus |B^{\alpha-\epsilon}|)$. For ε sufficiently small it holds that D_ϵ is a topological ball (of the same dimension as D), and $|B^{\alpha+\epsilon}|$ and $|B^{\alpha-\epsilon}| \cup D_\epsilon$ are homotopy equivalent.*

The main theorem of this section is the following. In its proof we use the symbol $\simeq$ to denote homotopy equivalence.

Theorem 4 *Let P be a finite set of weighted points in $\mathbb{R}^d$. For every $\alpha \geq 0$ the α-shape $|K^\alpha(P)|$ and the flow-shape $|F^\alpha(P)|$ are homotopy equivalent.*

Proof. By Theorem 1 $|B^\alpha(P)|$ and $|K^\alpha(P)|$ are homotopy equivalent for all $\alpha \geq 0$. It is therefore sufficient to prove the homotopy equivalence of $|B^\alpha(P)|$ and $|F^\alpha(P)|$. The homotopy type of both $|B^\alpha(P)|$ and $|F^\alpha(P)|$ changes only at critical levels of the distance function h. We therefore prove their homotopy equivalence by induction over the critical levels of the distance function. We may assume without loss of generality that all points have non-positive weights. For $\alpha = 0$ we have $|B^\alpha(P)| = P_0 = |F^\alpha(P)|$, where P_0 is the set of points of weight 0.

Let $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n$ be the critical levels of h and assume that $|B^\alpha(P)|$ and $|F^\alpha(P)|$ are homotopy equivalent for all $\alpha \leq \alpha_{i-1}+\epsilon$, where $\epsilon > 0$ has to satisfy the following:

(i) The ϵ-neighborhood of any critical level α_j does not contain any other critical level.

(ii) For any critical point x of h it can be seen from the recursive construction of the stable manifold S of x in the algorithm Inflow that there exists $\epsilon_x > 0$ such that $D_{\epsilon_x} \subset S$, where $D_{\epsilon_x} = D \setminus |B^{\alpha-\epsilon_x}|$. We need that $0 < \epsilon < \min\{\epsilon_x \,|\, x \text{ critical point}\}$. Note that there can be only finitely many critical points of h by Observation 1.1.

In the induction step, we want to prove the homotopy equivalence of $|B^\alpha(P)|$ and $|F^\alpha(P)|$ for all $\alpha \leq \alpha_i + \epsilon$.

Let $x \in h^{-1}(\alpha_i)$ be a critical point of h. Without loss of generality x is the only critical point in $h^{-1}(\alpha_i)$. If $x \in P$ then the equivalence is straightforward: in both complexes a new component homotopy equivalent to a point appears. Assume $x \notin P$. Suppose $|F^{\alpha_i}(P)| \setminus D_\epsilon \simeq |F^{\alpha_{i-1}}(P)|$, where D_ϵ is as in Theorem 2. We have

$$\begin{aligned} |F^{\alpha_i}(P)| \setminus D_\epsilon &\simeq |F^{\alpha_{i-1}}(P)| = |F^{\alpha_{i-1}+\varepsilon}(P)| \\ &\simeq |B^{\alpha_{i-1}+\varepsilon}(P)| \simeq |B^{\alpha_i-\varepsilon}(P)|, \end{aligned}$$

where equality follows from the fact that flow shapes only change at the critical levels, the homotopy equivalences in the second line follow from the induction hypothesis, Theorem 2 and our assumption on ϵ. Since by construction the boundary of D_ϵ is contained in both the boundary of $|F^{\alpha_i}(P)| \setminus D_\epsilon$ and in the boundary of $|B^{\alpha_i-\varepsilon}(P)|$ we get a homotopy equivalence, $|F^{\alpha_i}(P)| = (|F^{\alpha_i}(P)| \setminus D_\epsilon) \cup D_\epsilon \simeq |B^{\alpha_i-\varepsilon}(P)| \cup D_\epsilon$. That is, we get using Lemma 3,

$$\begin{aligned} |B^{\alpha_i+\varepsilon}(P)| &\simeq |B^{\alpha_i-\varepsilon}(P)| \cup D_\epsilon \\ &\simeq |F^{\alpha_i}(P)| = |F^{\alpha_i+\varepsilon}(P)|. \end{aligned}$$

Hence the union of balls $|B^{\alpha_i+\varepsilon}(P)|$ and the flow shape $|F^{\alpha_i+\varepsilon}(P)|$ have the same homotopy type at level $\alpha_i+\varepsilon$. Since we know that the flow shape and the union of balls can change their homotopy type in the interval $[\alpha_i - \varepsilon, \alpha_i + \varepsilon]$ only at the critical level α_i we have that they are homotopy equivalent for all $0 \leq \alpha \leq \alpha_i + \varepsilon$.

That leaves us to show that $|F^{\alpha_i}(P)| \setminus D_\epsilon \simeq |F^{\alpha_{i-1}}(P)|$ holds. Let S be the closure of the stable manifold of x and let ∂S be its relative boundary. Considering the local situation the above can be restated as $S \setminus D_\epsilon \simeq \partial S$. We prove this by giving a deformation retract using the structure of S inherently described by the algorithm Inflow. The algorithm processes a sequence of Voronoi objects (its second argument). For the proof we rearrange the order of processing these Voronoi objects in a breadth first manner: We collect all flow from higher dimensional Voronoi objects before we collect flow from the boundary of a Voronoi object.

This gives us a hierarchy as follows: we start with a point (the critical point) and a Voronoi object V it is contained in. Then we process all Voronoi objects that contain V together with some higher dimensional polytope. Whenever we process new flow coming from the boundaries of previously processed Voronoi cells a new step in the hierarchy starts. Assume that we have m steps in the hierarchy and let $S_j, j = 1, \ldots, m$ be the relative interior of the part of the stable manifold S of x that has been constructed after finishing step j of the hierarchy.

We first show that $S \setminus D_\epsilon \simeq S \setminus S_1$. To this end consider the boundary ∂S_1 of S_1. It has the structure of a polyhedral complex and is visible from the critical point. With D_ϵ removed we can therefore retract to the boundary of S_1. Next we want to show that $S \setminus S_1 \simeq S \setminus S_2$. For this we consider a cell P of $\partial S_1 \cap S_2$ together with the corresponding Voronoi object V. The proof proceeds by showing that the area of flow onto P from higher-dimensional Voronoi objects can be retraced starting at P, the details are left out here because of space constraints. From this we get $S \setminus D_\epsilon \simeq S \setminus S_2$. In the same way we can continue with the remaining steps in our hierarchy and finally we get $S \setminus D_\epsilon \simeq S \setminus S_m = \partial S$. □

References

[1] T. K. Dey, J. Giesen, and S. Goswami. Shape segmentation and matching with flow discretization. In *Proc. 8th Intern. Workshop on Algorithms and Data Structures*, pages 25–36, 2003.

[2] T. K. Dey, J. Giesen, and M. John. Alpha-shapes and flow shapes are homotopy equivalent. In *Proc. 35th Symp. Theory of Computing*, pages 493–501. 2003.

[3] H. Edelsbrunner. The union of balls and its dual shape. *Discrete Computational Geometry*, 13:415–440, 1995.

[4] J. Giesen and M. John. Computing the weighted flow complex. In *Proc. 8th International Fall Workshop Vision, Modeling, and Visualization*, pages 235–243, 2003.

[5] J. Giesen and M. John. The flow complex: A data structure for geometric modeling. In *Proc. 14th ACM-SIAM Sympos. Discr. Algorithms*, pages 285–294, 2003.

[6] K. Grove. Critical point theory for distance functions. In *Differential geometry: Riemannian geometry*, volume 54 of *Proc. Sympos. Pure Math.*, pages 357–385. Amer. Math. Soc., 1993.

[7] M. John. *Flow complexes : structure, algorithms and applications.* PhD thesis, Technische Wissenschaften ETH Zürich, 2003.

[8] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu. *Spatial Tesselations: Concepts and Applications of Voronoi Diagrams.* Wiley, second edition, 2000.

[9] D. Siersma. Voronoi diagrams and Morse theory of the distance function. In *Geometry in Present Day Science*, pages 187–208. World Scientific, 1999.