Minimizing the Total Absolute Gaussian Curvature in a Terrain is Hard*

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Abstract

We show that re-triangulating a terrain in order to minimize its total absolute Gaussian curvature, under the constraint that we fix the vertex set and boundary of the terrain, is NP-hard.

1 Introduction

In contrast to the total Gaussian curvature of a surface embedded in $\mathbb{R}^3$, which is a topological invariant of the surface (Gauss-Bonnet theorem), the total absolute Gaussian curvature encodes a lot of information about the embedding. The first to realize that the total absolute Gaussian curvature might be important in surface processing were Alboul and van Damme [1]. They suggested to post-process a surface mesh (polyhedral surface) such that the topology and the vertex set of the polyhedral surface is kept while the total absolute Gaussian curvature is minimized. In order to minimize the total absolute curvature they use a simple flip heuristic for which they can prove that it gives the optimal result if the vertex set is in convex position, which in this case is the convex hull of the vertex set. In the non-convex case, which from the application point of view is the interesting one, it is easy to see that the heuristic can get stuck in a local minimum. Nevertheless, even in the non-convex case the heuristic improves the visual appearance of the mesh significantly, see Figure 1.

It remained an open question, whether an efficient algorithm exists which always finds the global minimum. Here we show that, at least in the case of terrains, minimizing the total absolute curvature is NP–hard.

2 Definitions and Motivation

Polyhedral surface. The object of our study are *polyhedral surfaces*, which are the geometric realizations of simplicial complexes in $\mathbb{R}^3$, whose underlying topological space is a surface with or without boundary. We refer to polyhedral surfaces without boundary also as closed surfaces.

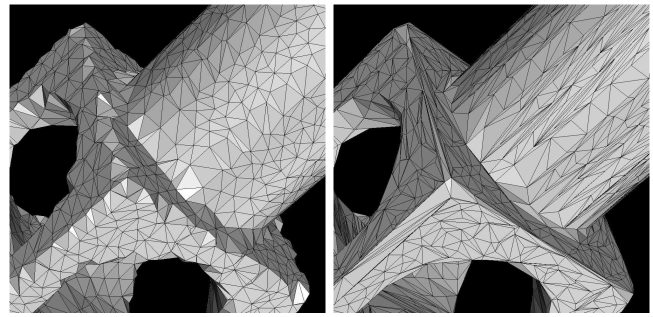

Figure 1: A triangle mesh before (left) and after (right) applying the flip heuristic to minimize the total absolute Gaussian curvature. Pictures courtesy of P. Gehr [3].

A terrain is a polyhedral surface with boundary whose vertices have two coordinates in a common hyperplane and the orthogonal projection of the surface onto this hyperplane is one-to-one.

Tight surface. A closed polyhedral surface is called *tight* if any hyperplane cuts it in at most two pieces (*two-piece-property*). Note that convex surfaces are always tight. Since a surface of higher genus, e.g., a torus, cannot be convex, the notion of tightness can be seen as a generalization of the concept of convexity to surfaces of higher genus.

Total absolute Gaussian curvature. The *Gaussian curvature* of a polyhedral surface S is defined at its vertices. Let v be a vertex of S. For any triangle T_i in S incident to v let α_i be the angle of T_i at v. The Gaussian curvature K_v of S at v is defined as $K_v = 2\pi - \sum_i \alpha_i$. For the total Gaussian curvature the famous Gauss-Bonnet theorem holds:

$$\sum_{v \in S} K_v \;=\; 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic of S. Also the *absolute Gaussian curvature* is defined at the vertices of S. For this we consider the *positive curvature* K_v^+ and *negative curvature* K_v^- at a vertex v of S. Let us first define the positive curvature. We distinguish two kinds of vertices v depending on whether S has a

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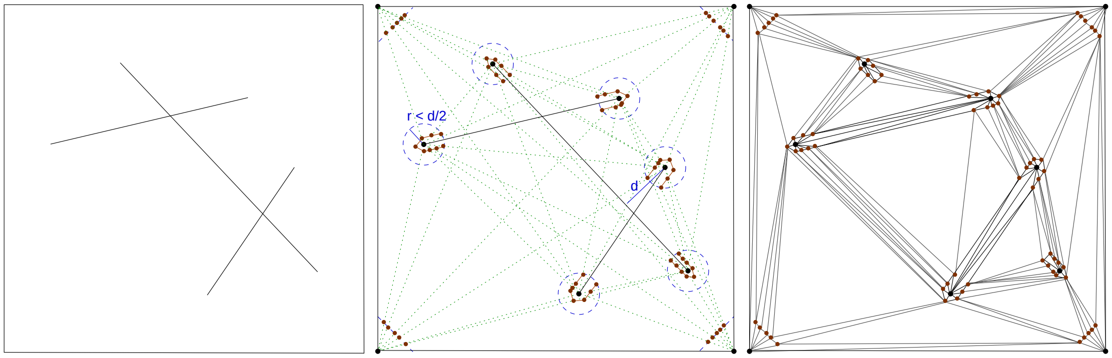

Figure 2: A set of line segments, construction of the terrain and re-triangulation of the terrain.

local supporting hyperplane at v or not. We say that S has a local supporting hyperplane at v if there exists a neighborhood U of v in S such that U is completely contained in one of the two closed half-spaces bounded by the hyperplane. If S does not have a local supporting hyperplane at v we set $K_v^+ = 0$. Otherwise, let T_j' be a triangle on the boundary of the convex hull of v and all vertices of S incident to v. Let β_j be the angle of T_j' at v. One defines $K_v^+ = 2\pi - \sum_j \beta_j$. The negative curvature at any vertex v of S is now defined as $K_v^- = K_v^+ - K_v$ and the absolute Gaussian curvature of S at v is defined as $|K_v| = K_v^+ + K_v^-$.

For polyhedral surfaces with boundary we define the curvatures only at interior vertices, i.e., at vertices which have a neighborhood homeomorphic to a disk.

Theorem 1 *The following are equivalent [5]:*

(1) S is a tight closed surface.

(2) The total absolute Gaussian curvature is the same as the total Gaussian curvature, i.e., $2\pi\chi(S)$.

In surface post-processing, as described in the introduction, the notion of tightness is a promising approach. Sharp edges of a model are often reconstructed in a jaggy pattern, as in Figure 1 (left). For such a jaggy pattern there is a hyperplane that cuts it into many pieces, i.e. the surface is very far from being tight. Hence one might try to compute the 'tightest' re-triangulation of the surface. This leads to two constrained optimization problems, the constraints in both cases are that S' shares the vertex set V with S and has the same topology as S (and boundary in the case of terrains):

(1) Compute S' that minimizes the maximum number of components it can be cut into by a hyperplane.

(2) Compute S' that minimizes the total absolute Gaussian curvature.

3 NP–Hardness

In this section we prove the NP-hardness of the problem of minimizing the total absolute Gaussian curvature of a terrain. We do so by giving a reduction from the *non-intersecting line segments problem*, which is known to be NP-hard, if the segments lie in at least three directions and any two parallel segments are disjoint [4].

Non-intersecting line segments. An instance of the non-intersecting line segments problem consists of a set of n line segments $L = \{l_1, \ldots, l_n\}$ whose endpoints have rational coordinates and a positive integer m. The question is whether there exists a subset $L' \subset L$ s.t. $|L'| \geq m$ and none of the segments in L' intersect.

Now we can state and prove the main theorem.

Theorem 2 *Minimizing the total absolute Gaussian curvature in a terrain is NP-hard.*

Proof. For the hardness proof we consider the decision problem: Given a terrain S and a number κ. Is there a terrain with the same vertex set as S that has total absolute Gaussian curvature less than κ? This problem is in NP since the total absolute Gaussian curvature for a terrain can be computed in polynomial time.

We give a reduction from the *non-intersecting line segments problem.* The reduction is similar to a reduction given in a proof showing that minimizing the number of minima in a terrain is hard [2].

Given an instance (L, m) of the non-intersecting line segments problem we have to compute in polynomial time a terrain S and a bound on the total Gaussian curvature κ of S such that there exists a re-triangulation of S (retaining the boundary and the vertex set of S) with curvature less than κ if and only if L has a subset of at least m pairwise non-intersecting line segments.

Construction of the terrain The idea of the construction is, that we use the endpoints of the segments as height points of our terrain and add further points at

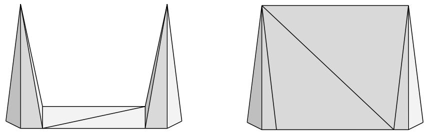

Figure 3: The curvature of connected height points (right) is much smaller than for not connected height points (left).

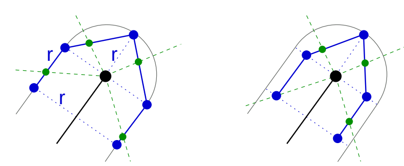

Figure 4: The horseshoe construction. A normal horseshoe (left) and a compressed horseshoe (right).

height 0 such that only height points of the same segment may be connected and this only if no height points of intersecting segments have been connected. At connected height points the Gaussian curvature is much smaller than at not connected height points, see Figure 3. In particular, it is smallest if we can connect as many segments as possible.

More explicitly, we construct S as follows (compare with Figure 2): Let $n := |L|$ and assume that all line segments in L lie in the quadrilateral Q with corner points $(0,0), (0,1), (1,0)$ and $(1,1)$ in the xy-plane. The vertex set of S contains all endpoints of line segments in L lifted to a height h in the z-direction, the value of h we will specify later. For each endpoint, we add $2n+3$ points on a small "horseshoe" around them in the xy-plane, which prevent the lifted endpoints of different segments to be connected. Note that these connections are only prevented because we demand that any valid re-triangulation of the point set has to be a terrain over the xy-plane. Each horshoe is a polyline with five vertices that all have rational coordiantes, see Figure 4. To compute the horseshoes, we first compute the minimal distance d between a segment and a non corresponding endpoint or the quadrilateral Q. Then we know that the if we "blow-up" all segments by a radius $r < d/2$, i.e. replace each segment by the Minkowski sum of itself with a ball of radius r, the "blown-up" segments do not intersect. The horseshoes are situated on the boundary of these "blown-up" segments. We place at most $2(n-1)$ points at the intersection of the horseshoe with lines from the segment endpoint to all non corresponding segment endpoints. We might need to "compress" the horseshoes such that all these lines intersect the horseshoe. If we compress one horseshoe we do so also with the other horseshoe of the segment. If a non corresponding endpoint is collinear with the segment and lies behind the other endpoint we can ignore it. Placing the points on the horseshoes can be done in polynomial time as there are only a polynomial number of distance computations necessary and the intersection point of two line segments whose endpoints have rational coordinates can be computed in time polynomial in the bit description length of the endpoints.

As fixed boundary points of the terrain S we choose the corner points of the quadrilateral Q. Around each corner point we put $2n$ points on a line segment that separates the corner points from all other points in the xy-plane. We do so to prevent the lifted endpoints of the segments to be connected to the corner points (similarly as with the points on the horseshoes). We do this because we do not consider the curvature at the boundary i.e. the corner points and therefore allow the corner points to be connected only to points at height 0.

Now we have defined the vertex set of S, which can be computed in polynomial time. It remains to triangulate this vertex set. We choose S to be the following triangulation: We connect all lifted segment endpoints to the points of their corresponding horseshoes. We connect the points on the horseshoe along the horseshoe and we connect the two endpoints of each horseshoe with each other. Finally we choose an arbitrary triangulation of the enclosing quadrilateral in the regions between the horseshoes. Note that the whole triangulation can be computed in polynomial time in the size of L.

Next we determine the value of κ depending on m: Putting a copy of the boundary quadrilateral below (in the z-direction) the original one and triangulating in the canonical way gives us for any terrain on the given point set a triangulated topological sphere. For any such sphere, the corner points of the two quadrilaterals (the boundary of the terrain and its copy) are convex vertices and the angles of triangles incident to such a corner point sum up to $3\pi/2$. That is, any such corner point v has $K_v = K_v^+ = \pi/2$. Furthermore, for any vertex v it holds

$$|K_v| = K_v^+ + K_v^- = K_v^+ + (K_v^+ - K_v) = 2K_v^+ - K_v$$

and $\sum_v K_v$ is constant for a topological sphere by the Gauss-Bonnet theorem. Hence minimizing the total absolute Gaussian curvature in terrains as we consider them here is equivalent to minimizing the total positive curvature. We choose $\kappa = (2n-m)2\pi$ as bound on the total positive curvature.

Finally we compute the value of the height h. We choose it such that we get sufficient lower bounds for the positive curvature both at a connected lifted segment endpoint v_1 and at a not connected lifted segment endpoint v_2. Let us first consider v_1. We have a trivial upper bound of π on the positive curvature at v_1, because we definitely have $\sum \beta_j \geq \pi$, where the β_j are as in the definition of the positive curvature. We can lower bound this curvature by the positive curvature of the apex of the half-cone, whose apex $(1, \frac{1}{2}, h)$ is connected with the points $(-\frac{1}{2}, 1, 0), (-\frac{1}{2}, 0, 0), (\frac{1}{2}, 0, 0), (\frac{3}{2}, 0, 0)$ and $(\frac{3}{2}, 1, 0)$ in the xy-plane as shown in Figure 5. Note that the triangle with vertices $(1, \frac{1}{2}, h), (-\frac{1}{2}, 1, 0)$ and $(\frac{3}{2}, 1, 0)$ does not belong to the half-cone.

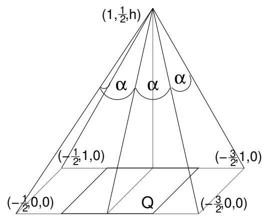

Figure 5: Bounding the positive curvature at a lifted segment endpoint

The positive curvature of this half-cone is $\pi - 4\alpha$, where α is the angle at the apex $(1, \frac{1}{2}, h)$ in its incident triangles. Using the law of cosine we get $\cos^2 \alpha = (h^2 + 1)/(h^2+2)$. Similarly we get a trivial upper bound of 2π and a lower bound of $2(\pi - 4\alpha)$ for the positive curvature at v_2. A simple calculation shows that if we choose $h > 4(2n - m + 1)/\pi$ then it holds $4\alpha < \pi/(2n - m + 1) =: \varepsilon$.

Finding a triangulation with small positive curvature Assume now that L has a subset L' of non-intersecting line segments of size at least m. We have to show that a re-triangulation of S exists whose total positive curvature is less than κ. For this we consider the re-triangulation where the lifted endpoints of all segments in the subset L' are connected and all other segments are not connected. Of course, the endpoints of the horse-shoes at the connected segment endpoints have to be reconnected, see Figure 2. We partition the vertex set of S in three parts, V_1 contains all segment endpoints that are connected with their matching endpoint, V_2 contains all segment endpoints that stay not connected and V_3 contains the remaining vertices that are not on the boundary. We then have

$$\begin{aligned}\sum_v K_v^+ &= \sum_{v \in V_1} K_v^+ + \sum_{v \in V_2} K_v^+ + \sum_{v \in V_3} K_v^+ \\ &\leq 2m\pi + 2(n-m)2\pi + 0 \\ &= (2n-m)2\pi \;=\; \kappa,\end{aligned}$$

where the bounds for the first two sums follow immediately from the trivial upper bounds for the curvatures at v_1 and v_2 that we derived earlier. The third sum equals 0 because all vertices in V_3 have positive curvature 0. Either they lie in the convex hull of their neighbors, which is planar, and therefore have positive curvature 0. Or they lie on a straight edge, e.g., in the relative interior of a strictly straight part of a horseshoe, and also have positive curvature 0.

This shows that if L has a subset L' of non-intersecting segments of size $|L'| \geq m$, then there exists a re-triangulation of S with total positive curvature less than $\kappa = (2n - m)2\pi$.

Finding a large subset of non-intersecting segments It remains to show that if there is a re-triangulation of the terrain S with total positive curvature not larger than $\kappa = (2n - m)2\pi$ then L has to have a subset of at least m non-intersecting segments. Let $2\hat{m}$ be the number of lifted segment endpoints that are connected in the triangulation. Then we can bound the total positive curvature of the re-triangulation from below as follows: Again we partition the vertex set of S in three parts, V_1 contains all lifted segment endpoints that are connected with their matching endpoint, V_2 contains all lifted segment endpoints that are not connected and V_3 contains the remaining vertices that are not on the boundary. Remember that by our construction a lifted segment endpoint can only be connected to its matching lifted endpoint and to no other lifted endpoint. Thus $\hat{m}$ is the number of segments whose lifted endpoints are connected and has to be an integer. Using the lower bounds on the positive curvature at v_1 and v_2 which we derived earlier and the trivial lower bound of 0 for all other vertices, we get

$$\begin{aligned}\sum_v K_v^+ &= \sum_{v \in V_1} K_v^+ + \sum_{v \in V_2} K_v^+ + \sum_{v \in V_3} K_v^+ \\ &\geq 2\hat{m}(\pi - \epsilon) + 2(n - \hat{m})2(\pi - \epsilon) + 0 \\ &= (2n - \hat{m})2(\pi - \epsilon).\end{aligned}$$

By assumption $\sum_v K_v^+ \leq \kappa = (2n-m)2\pi$ and therefore $(2n - m)2\pi \geq (2n - \hat{m})2(\pi - \epsilon)$. Solving for $\hat{m}$ we get

$$\hat{m} \geq m - \frac{\epsilon(2n - m)}{\pi - \epsilon}.$$

For $\epsilon < \pi/(2n-m+1)$ the last term is smaller than 1 and as $\hat{m}$ is an integer this yields $\hat{m} \geq m$. Thus L contains a subset of at least m non-intersecting segments. □

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