On the Pagenumber of Bipartite Orders

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Abstract

The pagenumber problem for ordered sets is known to be NP-complete, even if the order of the elements on the spine is fixed. In this paper, we investigate this problem for some classes of ordered sets. We provide an efficient algorithm for drawing bipartite interval orders in the minimum number of pages needed. We also give an upper bound for the pagenumber of general bipartite ordered sets and the pagenumber of the complete multipartite ordered sets with length four and five.

1 Introduction

A large number of relevant problems in different domains can be formulated as graph layout problems (see Diaz et al. [2] for a survey). A book embedding (or stack layout) of a graph G consists of an embedding of its nodes along the spine of a book and embeddings of its edges on pages so that edges embedded on the same page do not intersect. The pagenumber of G, $page(G)$, is the minimum number of pages needed, taken over all permutations on the vertices of G. Applications of stack layouts include sorting permutations, fault tolerant VLSI design, complexity theory, compact graph encodings, compact routing tables, and graph drawing. Nowakowski and Parker [6] were the first to introduce the pagenumber of an ordered set. It is the stacknumber of an ordered set's Hasse diagram viewed as a directed graph. In a book embedding for an ordered set P, the vertices of P are embedded on the spine of the book to form a linear extension of P. Most of the known results relate to classes of ordered sets with a pagenumber two, and even the question regarding a general characterization of ordered sets with pagenumber two is still open. Several questions on the pagenumber problem are shown to be NP-complete: if the order of nodes on the spine is fixed, whether an ordered set can be embedded in 6-pages, computing the page number of a bipartite order, etc. (see[5] for an extensive review). Perhaps the only challenging class where a precise solution was found is the class of series-parallel planar ordered sets: Alzohairi and Rival [1] showed that the pagenumber of any series-parallel planar ordered set is at most two. Giacomo et al. [4] presented a better algorithm (linear time) to embed series-parallel planar ordered sets into two pages. In this paper, we compute the page number in some restricted classes of ordered sets. We prove that the pagenumber of a bipartite interval order P is equal to the maximum pagenumber of a complete suborder of P. We use a technique that relies on easily identifying complete suborders within a given bipartite interval order. An algorithm for finding the pagenumber of bipartite interval orders is deduced. The strategy we used for bipartite interval order turns out to be helpful in finding an upper bound for the pagenumber of bipartite ordered sets in general.

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2 Definitions

Let P be an ordered set and let x be in P. The set of successors (resp. predecessors) of x in P, denoted *Succ(x)* (resp. *Pred(x)*), is the set of all elements y in P such that $x \leq y$ (resp. $x \geq y$). An interval representation of an ordered set $(P,<)$ is a function that assigns to each element u in P an interval on the real line I_u such that $u < v$ if and only if each point of I_u is less than every point in I_v. If an ordered set $(P,<)$ has an interval representation, then we call $(P,<)$ an *interval order* (see example Figure 1). Interval orders have very nice characterizations which give more information about the structure and make them more understandable [3]. An ordered set P is an interval order if and only if P does not contain a $2 \otimes 2$ as induced suborder, that is, a subset $\{u, v, x, y\}$ of P with $u < v$ and $x < y$ are the only comparabilities among these elements. In any interval order P the following important condition also holds: the sets of predecessors (as well as the sets of successors) are linearly ordered with respect to inclusion. That is, for all $x, y \in P$, either *Pred(x)* $\subseteq$ *Pred(y)* or *Pred(x)*$\supseteq$ *Pred(y)*.

3 The Pagenumber of Bipartite Interval Orders

Theorem 1 *The pagenumber of a bipartite interval order is equal to the maximum pagenumber of complete suborders of P.*

First, note that for a bipartite ordered set P, $page(P') \leq page(P)$ if P' is a suborder of P. In order to prove Theorem 1, we need a sequence of lemmas. Let $P = (M, N)$ be a bipartite interval order with a set

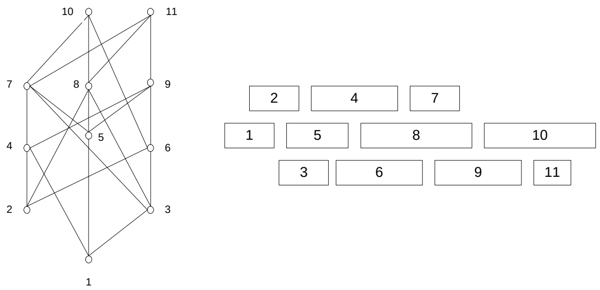

Figure 1: An interval order its interval representation.

M of minimal elements of size m and a set N of maximal elements of size n. Let $M = (m_1, m_2, \ldots, m_m)$ be the list of minimal elements of P arranged in a decreasing order with respect to the inclusion relation of the sets of successors, i.e. $Succ(m_1) \supseteq Succ(m_2) \supseteq \ldots \supseteq Succ(m_m)$. Let $N = (n_1, n_2, \ldots, n_n)$ be the list of maximal elements of P arranged in decreasing order with respect to the inclusion relation of the sets of predecessors, i.e. $Pred(n_1) \supseteq Pred(n_2) \supseteq \ldots \supseteq Pred(n_n)$.

Lemma 2 *Let $P = (M, N)$ be a bipartite interval order, and let $P' = (M', N')$ be a complete suborder of P. Then there exists i and j such that $M' \subseteq \{m_1, m_2, \cdots, m_i\}$ and $N' \subseteq \{n_1, n_2, \cdots, n_j\}$ and $(\{m_1, m_2, \cdots, m_i\}, \{n_1, n_2, \cdots, n_j\})$ is a complete bipartite suborder of P.*

Proof. Let j be the maximum key such that $n_j \in N \cap N'$ (j exists since $N' \subseteq N$). Since $Pred(n_j) \subseteq Pred(n_k)$ for every $k \leq j$ then $(M', \{n_1, n_2, \cdots, n_j\})$ is a complete suborder of P. Likewise, let i be the maximum key such that $m_i \in M \cap M'$(i exists since $M' \subseteq M$). Since $Succ(m_i) \subseteq Succ(m_k)$ for every $k \leq i$, $(\{m_1, m_2, \cdots, m_i\}, \{n_1, n2, \cdots, n_j\})$ is a complete bipartite suborder of P. Moreover it is obvious that $M' \subseteq \{m_1, m_2, \cdots, m_i\}$ and $N' \subseteq \{n_1, n_2, \cdots, nj\}$ $\Box$

Lemma 2 shows that in order to find a complete bipartite suborder of P with the maximum pagenumber, it is sufficient to look at those ones with the structure $(\{m_1, m_2, \cdots, m_i\}, \{n_1, n_2, \cdots, n_j\})$, for some i and j. Henceforth, $P(i, j)$ will be used to refer to these complete bipartite suborders.

Lemma 3 *Let $P = (M, N)$ be a bipartite interval ordered set, and let $P_1 = P(i, j)$ be a complete suborder of P with a maximal pagenumber.*

If $j > i$ then for every $k, l > i$, n_k is non comparable to m_l in P.

If $j < i$ then for every $k, l > j$, n_k is non comparable to m_l in P.

Proof. Suppose that $j > i$, therefore $page(P_1) = min\{i, j\} = i$. Suppose that $n_k > m_l$ in P for some key k and $l > i$. Since $Succ(m_l) \subseteq Succ(m_r)$ for every $r \leq l$, $Succ(m_{i+1})$ will contain $\{n_1, n_2, \cdots, n_j\}$ and therefore $P_2 = P(i+1, j) = (\{m_1, m_2, \cdots, m_i, m_{i+1}\}, \{n_1, n_2, \cdots, n_j\})$ is a complete suborder of P and where $page(P_2) = min\ \{i+1, j\} = i+1 > page(P_1)$ [Since $i < j$]. This contradicts the choice of P_1. The same argument will apply if $j < i$. $\Box$

Proof. [of Theorem 1] Consider a complete suborder $P' = P(i, j)$ of P with a maximum pagenumber. We prove that the bipartite interval ordered set P can also be embedded in $page(P')$ which is equal to the minimum of the two levels i.e. i, j. The optimal layout of P is obtained using the following linear extension L of P:

$$m_m < m_{m-1} < \cdots < m_1 < n_n < n_{n-1} < \cdots < n_1$$

Without loss of generality, we may assume that $i < j$. We will use the first page to draw all covering relations of m_1 and of n_1. Clearly, we can fit all these covering relations within the first page. This is possible because the covering relation between m_1 and n_1 will leave an empty hole that we could utilize to embed the relations between n_1 and the elements in M (see Figure 2 (a)). We continue to draw the covering relations between m_2 and n_2 in the second page and so on.

Overall, we will be able to draw all of the left cover relations of the elements m_k and n_k on the k^{th} page for $k \leq i$. Lemma 3 guarantees that all the covering relations of P will be drawn in one of the pages after i iterations (see Figure 2(b)). $\Box$

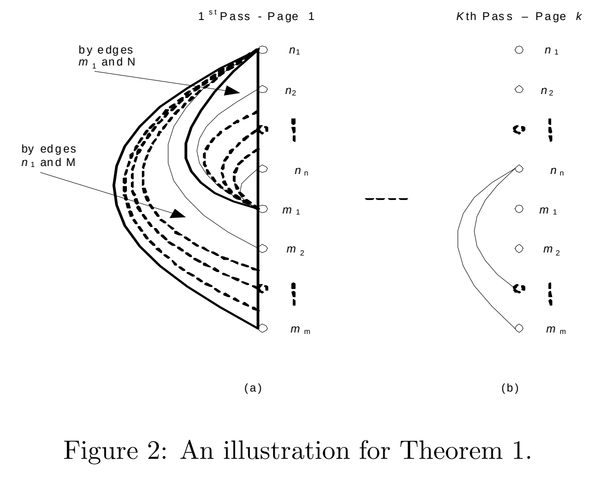

Figure 2: An illustration for Theorem 1.

Complexity of the Algorithm: the drawing algorithm implied from Theorem 1 consists of two parts: the preprocessing (sorting the list of minimal elements of P in a decreasing order with respect to the inclusion relation of the sets of successors and sorting the list of maximal elements of P in decreasing order with respect to the inclusion relation of the sets of predecessors) and drawing stage. It is easy to see that both stages of the algorithm have a complexity of $O(n^2)$ where n is the number of elements in P.

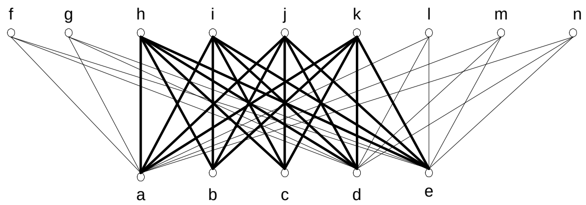

Figure 3: A bipartite interval order.

In Figure 3, the largest complete suborder is $P'(5, 4)$ which can can be embedded in four pages. Likewise, the bipartite interval order can be embedded in four pages with the following linear extension $L : b < c < e < a < d < n < m < l < g < f < k < j < i < h$, as shown in Figure 4.

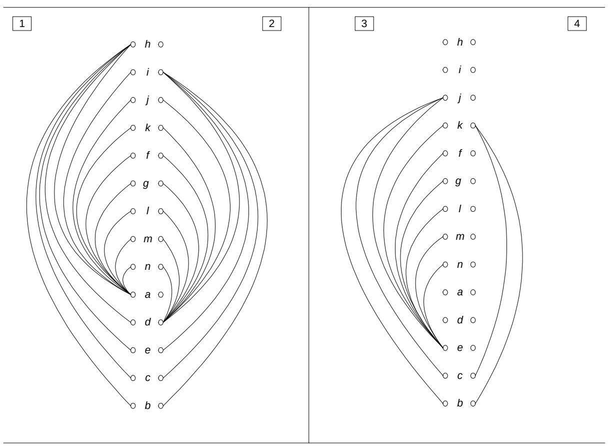

Figure 4: Four pages embedding of the order of Figure 3.

The result of Theorem 1 cannot be extended to the case of n-partite interval orders. We do not see a generalization for the n-partite interval order and not even for the tripartite case. For instance, Figure 5 illustrates a tripartite interval order P which has $page(P) = 4$ although there are no complete tripartite suborders with pagenumber larger than 3. P' is a complete tripartite suborder obtained from P with a maximum pagenumber.

4 An Upper Bound for the Pagenumber of Bipartite Ordered Sets

The strategy we used in the last section turns out to be helpful in finding an upper bound for the pagenumber of a general bipartite ordered sets. Therefore, we can utilize a similar strategy to assist us in finding an upper bound for the pagenumber of bipartite ordered sets.

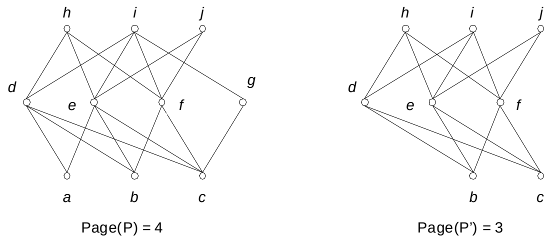

Figure 5: Interval order P with the complete multipartite suborder P' of P.

We define a *zig-zag* Z of length $2n$ in P as a partition into disjoint subsets of $M = M_1, M_2, \cdots, M_n$ and $N = N_1, N_2, \cdots, N_n$ such that

$Succ(M_i) \subseteq N_i \cup N_{i+1}$ for $1 \leq i < n$ and $Succ(M_n) \subseteq N_n$ and

$Pred(N_i) \subseteq M_{i-1} \cup M_i$ for $1 < i \leq n$ and $Pred(N_1) \subseteq M_1$.

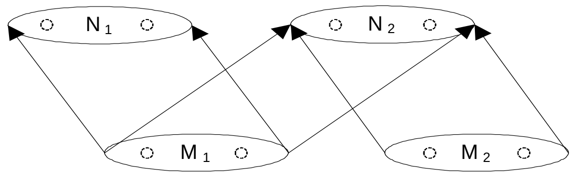

Figure 6: A zig-zag of length four.

Theorem 4 *Let $P = (M, N)$ be a bipartite ordered set. Let $Z = M_1, M_2, N_1, N_2$, be a zig-zag of length 4 that covers P. Then, the pagenumber of P is bounded by the maximum value of $|M1|$ and $|N2|$.*

Proof. Let us enumerate all the elements of the disjoint sets of the *zig-zag* namely, M_1, M_2, N_1 and N_2 in random orders: $M_1 : m_{1,1} < m_{1,2} < \cdots < m_{1,m}, M_2 : m_{2,1} < m_{2,2} < \cdots < m_{2,n}, N_1 : n_{1,1} < n_{1,2} < \cdots < n_{1,p}, N_2 : n_{2,1} < n_{2,2} < \cdots < n_{2,q}$. Let us assume L is a linear extension of P obtained by enumerating the elements of P using the following ranking order:

$$\begin{aligned} & m_{1,1} < m_{1,2} < \cdots < m_{1,m} \\ & < m_{2,1} < m_{2,2} < \cdots < m_{2,n} \\ & < n_{2,1} < n_{2,2} < \cdots < n_{2,q} \\ & < n_{1,1} < n_{1,2} < \cdots < n_{1,p} \end{aligned}$$

We can utilize the first page to draw all covering relations of $m_{1,1}$ and $n_{2,1}$ as follows. It is possible to accommodate all these covering relations within the same page (1^{st} page) since area between $m_{1,1}$ and $n_{2,1}$ is not used (we may have a covering relation between these two vertices) and thus could be used to draw the relations between $n_{2,1}$ and the elements in M.

After k iterations such that $k \leq max\{|M_1|, |N_2|\}$, we will be able to embed all of the remaining covering relations of the elements $m_{1,k}$ and $n_{2,k}$ in the k^{th} page.

After $max\{|M1|, |N2|\}$ iterations, all the covering relations of P will be drawn in one of the pages because there are no covering relations between the elements of N_1 and the elements of M_1.

$\square$

Although we have been successful to find an upper bound for the complete multipartite ordered sets of length four and five, we cannot see an easy way to generate a recursive formula that can be generalized to compute the pagenumber of complete multipartite ordered set with length n.

5 Pagenumber of Complete Multipartite Ordered Sets

In this last section, we give an approximation of the pagenumber of complete multipartite ordered sets of length four and five. It is not an easy task to find the pagenumber for even a special class of complete multipartite ordered sets. Recall that $page(P) = min\{|L1|, |L2|\}$ for complete bipartite ordered sets and $page(P) = min\{|L2|, |L1| + |L3|\}$ for tripartite ordered sets [7]. A similar approach is used in both cases to obtain optimal embedding. This approach finds a vertex cover C with a minimum size and then uses a separate page to embed all covering relations of a single element of C. We can approximate the pagenumber of a complete multipartite ordered set P by computing its minimal vertex cover. The covering graph of P can be viewed as a bipartite graph G. We can observe that every vertex cover of P corresponds to a vertex cover in G. We will omit the proofs of Theorem 5 and 6. It is based on the idea of limiting the embeddings to the ones related to minimal vertex coverings of the ordered set.

Theorem 5 *Let $P = (L_1, L_2, L_3, L_4)$ be a complete multipartite ordered set of length 4. Then, the pagenumber of P is bounded by the minimum of* $\{|L_1| + |L_3|, |L_2| + |L_4|, |L_2| + |L_3| - 1\}$.

Theorem 6 *Let $P = (L_1, L_2, L_3, L_4, L_5)$ be a complete multipartite ordered set of length 5. Then the pagenumber of P is bounded by the minimum of $|L_2| + |L_4|$ and $|L_3| + max(min(|L_1|, |L_2| - 1), min(|L_4| - 1, |L_5|))$.*

6 Conclusion

In this paper, we look at the problem of the pagenumber of bipartite ordered sets. We give a polynomial algorithm finding the exact pagenumber of bipartite interval orders. We also give an upper bound for general bipartite orders, and multipartite orders of length four and five.

Our solution for bipartite interval orders does not however extend to n-partites interval order, for which the question remains open.

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