Morphing Polyhedra Preserving Face Normals: A Counterexample

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Abstract

Two simple polyhedra P and Q (not necessarily convex) are said to be *parallel* if they share the same edge graph G and each face of P has the same normal as the corresponding face in Q. Parallel polyhedra P and Q admit a *parallel morph* if the vertices can be moved in a continuous manner taking us from P to Q, such that at all times the intermediate polyhedron determined by the vertex configuration and graph G is both simple, and parallel with P (and Q). In this note, we describe a pair of parallel orthogonal genus-0 polyhedra that do not admit a parallel morph.

1 Introduction

Assume we are given two straight-line drawings P and Q in $\mathbb{R}^d$ that are combinatorially equivalent (i.e., they represent the same graph). We say that P and Q are *parallel* drawings if corresponding edges have the same slope. A *parallel morph* is a continuous transformation between parallel drawings, such that every intermediate drawing is simple and is parallel with the original drawing. We extend this idea to polyhedra. Two polyhedra P and Q are called parallel if they share the same edge graph and if for every face in P, the corresponding face in Q has the same (unit) normal. Then, as is the case with drawings, a parallel morph between parallel polyhedra is a continuous transformation such that all intermediate polyhedra are both simple and parallel with P and Q.

We have been investigating the existence of parallel morphs, and the complexity of finding them for various classes of graphs and polyhedra [1, 2]. These questions are about connectivity within a parallel family of polygons/graphs/polyhedra: Can we go from one member of the family to any other via continuous changes that keep us within the family? Previous results are existence results for the case of planar cycles: Guibas et al. [9] and independently, Grenander et al. [8] prove that there is a parallel morph between any two parallel simple polygons in the plane. Before that Thomassen [15] had proved this result for the special case of orthogonal polygons (where each edge is aligned with one of the axes).

We have explored algorithmic issues for the case of more general graphs in the plane. We show [13] that for any pair of parallel orthogonal drawings of a graph, a parallel morph always exists and can be computed in time polynomial in the complexity of the graph. However, for non-orthogonal drawings the existence result fails and we show that the decision problem becomes NP-hard.

In three dimensions, even cycles present an interesting challenge. For one thing, cycles may be "knotted" in different ways, precluding the possibility of a continuous morph altogether. In [1] we show that restricting to cycles representing the trivial knot is not enough to guarantee a parallel morph even for the case of orthogonal cycles.

Although cycles in 3D seem complicated, we had hopes that polyhedra would be simpler. In fact, one of our conjectures for cycles was that they would be easy to morph if we could embed them on parallel genus-0 polyhedra. Unfortunately, we can use our NP-hardness result for graph drawings in the plane to prove the NP-hardness of deciding whether there exits a parallel morph between parallel genus-0 polyhedra. We still had hopes for orthogonal polyhedra—but the purpose of this brief note is to show a pair of parallel orthogonal genus-0 polyhedra that do not admit a parallel morph. In our construction we start with unmorphable cycles and then embed them on polyhedra which are thus unmorphable, since a parallel morph of the polyhedra would provide a parallel morph of the cycles.

Transforming one geometric configuration to another while maintaining some geometric structure is a broad topic with a rich background. It includes problems of morphing, motion planning, folding, linkage reconfiguration, rigidity theory, knot theory, etc. In the remainder of this section we mention some of the background relevant to our problem.

"Morphing" is a popular topic in graphics, but we use the term in a narrower sense: we assume

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that the correspondence between the source configuration and the target configuration is given, and we wish to maintain simplicity (the property that the configuration is non-self-intersecting).

Cairns in 1944 [3] showed that there is a non-intersecting morph from any planar triangulation to any isomorphic one with the same fixed triangle as a boundary. Thomassen [15] strengthened this to straight line drawings of planar graphs. He also considered preserving other geometric properties: he showed that convexity of faces can be preserved during such a morph; and he showed that edge directions can be preserved for the special case of orthogonal cycles. Floater and Gotsman [6] gave an entirely different approach to non-intersecting morphs of planar straight line drawings based on Tutte's graph embedding method. This was further explored by Gotsman and Surazhsky [7]. See also [5].

This paper is about parallel morphing, where we wish to maintain simplicity and the directions of the edges. A related topic is that of linkage reconfiguration; a form of morphing where simplicity and the *lengths* of the edges must be maintained. This is possible for limited classes of graphs: between any two simple chains/cycles in the plane with corresponding edges of the same length, there is a transformation that preserves simplicity and edge lengths [4, 14].

Finally we briefly mention the connections of our work to rigidity theory and parallel redrawings. More detail can be found in [1]. Preserving edge directions but not simplicity leads to the problem of *parallel redrawing* of graphs [12]. This turns out to be directly related to questions in rigidity theory where edge lengths but not simplicity are preserved. Curiously, the strong duality between parallel redrawings and rigidity theory falls apart when simplicity must be maintained, and the answers to linkage reconfiguration problems and parallel morphing questions do not appear related.

In the remainder of this section we describe the connection to knot theory. A *knot* is defined as a closed, non-self-intersecting curve embedded in $\mathbb{R}^3$. Two knots are equivalent if one knot can be transformed to the other by continuous deformed without self-intersection. Deciding whether two knots are equivalent is a central problem of *knot theory* (see [11] for an introduction).

The complexity of deciding knot equivalence has not yet been completely determined. The problem is in PSPACE [10]. A related problem, that of deciding whether a knot can be deformed to lie in a plane is in NP. There exist algorithms for both of these problems with running times that are exponential with respect to the number of crossings in an orthogonal projection of the knot(s) [10].

Suppose that we are given non-self-intersecting parallel drawings of a cycle graph in $\mathbb{R}^3$. Each drawing is a closed non-self-intersecting curve (i.e., a knot). If the drawings admit a parallel morph then they correspond to equivalent knots. However, the converse does not always hold [1].

2 Definitions

Let (V, E) be an undirected graph with vertex set V and edge set E. Let $p : V \to \mathbb{R}^d$ where d is a positive integer. We say that the triple $P = (V, E, p)$ is a *drawing* of graph (V, E) in $\mathbb{R}^d$, where each edge $(u, v) \in E$ is the straight-line segment between $p(u)$ and $p(v)$. A drawing that is not self-intersecting is called *simple*. Two drawings $P = (V, E, p)$ and $Q = (V, E, q)$ of graph (V, E) are called *parallel* if for each edge $(u, v) \in E$, there exists some $\lambda > 0$ such that $p(u) - p(v) = \lambda(q(u) - q(v))$. A *parallel morph* between parallel drawings P and Q is a continuously changing family of drawings $R(t) = (V, E, r^t)$ such that $R(0) = P$, $R(1) = Q$, and for all $t \in [0, 1]$, $r^t : V \to \mathbb{R}^d$ determines a simple drawing $R(t)$ that is parallel with P and Q.

Let P' and Q' be polyhedra in $\mathbb{R}^3$ with the same edge-graph (V, E). We say that P' and Q' are *parallel* if the two drawings of (V, E)—determined by vertex coordinates in each of P' and Q'—are parallel. Parallel polyhedra have equal unit normals on corresponding faces. A *parallel morph* $R'(t)$ between parallel polyhedra P' and Q' is a continuously changing family of polyhedra, such that for all $t \in [0, 1]$, $R(t)$ is both simple and parallel with P' and Q'. A drawing/polyhedron is *orthogonal* if each edge is parallel with one of the axes. An *orthodisk* is a closed region of the surface of an orthogonal polyhedron that is topologically equivalent to a closed disk, and whose boundary is an orthogonal drawing.

3 Main Result

In this section, we show that there exist parallel orthodisks that do not admit a parallel morph. We present parallel orthogonal drawings—topologically equivalent to the trivial knot—that do not admit a parallel morph. These drawings are augmented (adding additional vertices, edges and faces) to form unmorphable parallel orthodisks. From parallel or-

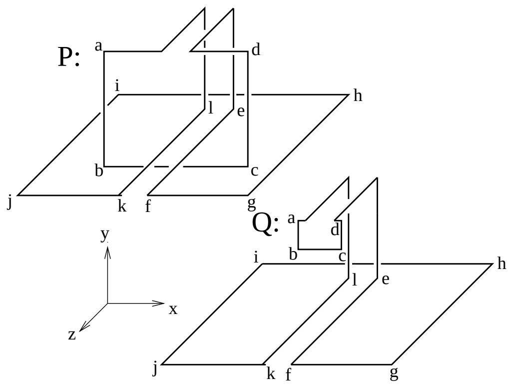

Figure 1: Parallel orthogonal drawings of a cycle that do not admit a parallel morph.

thodisks, it is trivial to construct parallel orthogonal genus-0 polyhedra.

Lemma 1 *The parallel drawings $P = (V, E, p)$ and $Q = (V, E, q)$ illustrated in Figure 1 do not admit a parallel morph.*

Proof. We introduce new notation. For each $v \in V$, let $p(v) = (p_x(v), p_y(v), p_z(v))$. Likewise for q. Notice that vertices $e, \ldots, l$ must lie in a common x-z plane in any drawing that is parallel with P and Q. We use the notation $p_y(e, \ldots, l)$ to denote the y-value of the plane containing vertices $e, \ldots, l$ in mapping p.

In P, $p_y(b, c) < p_y(e, \ldots, l)$ while in Q, $q_y(b, c) > q_y(e, \ldots, l)$. For the sake of contradiction, assume that there exists a parallel morph $R(t) = (V, E, r^t)$ from $P = R(0)$ to $Q = R(1)$. There must exist some $t \in [0, 1]$ such that $r^t_y(b, c) = r^t_y(e, \ldots, l)$. Let t_0 denote the smallest t for which equality holds.

In $R(t_0)$, b and c lie in the same x-z plane as do $e, \ldots, l$. However, before all these vertices can be made coplanar, the edge (b, c) must be moved to a position that does not overlap with edges (e, f) and (k, l) with respect to x and z axes. We will show that this cannot happen in any $R(t)$, where $t < t_0$. Therefore, there exists no parallel morph between P and Q.

Observe, for all drawings $R(t)$ where $t \in [0, 1]$, the following inequality holds:

$$\max(r^t_z(e), r^t_z(l)) < r^t_z(a, b, c, d) \tag{1}$$

Let us restrict our attention to drawings $R(t)$ where $t < t_0$. By definition, when $t < t_0$

$$r^t_y(b, c) < r^t_y(e, \ldots, l) < \min(r^t_y(a), r^t_y(d)) \tag{2}$$

Therefore, for all $t < t_0$ edge (a, b) in drawing $R(t)$ intersects the x-z plane through $r^t_y(e, \ldots, l)$ at some point. Let $\alpha^t = (\alpha^t_x, \alpha^t_y, \alpha^t_z)$ denote this point. Observe that $\alpha^t_z = r^t_z(a, b, c, d)$. Thus, by Equation 1, α^t_z must be larger than $r^t_z(l)$. Since the path of α must be continuous, and remains in the same x-z plane as $e, \ldots, l$ for $t < t_0$, we can bound α^t by the following.

$$r^t_z(l) < \alpha^t_z < r^t_z(j, k) \tag{3}$$

and

$$r^t_x(j, i) < \alpha^t_x < r^t_x(k, l) \tag{4}$$

Symmetrically, let $\beta^t = (\beta^t_x, \beta^t_y, \beta^t_z)$ where $t < t_0$ denote the point of intersection in $R(t)$ between edge (c, d) and the x-z plane through $r^t_y(e, \ldots, l)$. Then,

$$r^t_z(e) < \beta^t_z < r^t_z(f, g) \tag{5}$$

and

$$r^t_x(e, f) < \beta^t_x < r^t_x(g, h) \tag{6}$$

Notice that $\alpha^t_z = \beta^t_z = r^t_z(a, b, c, d)$, where $t < t_0$. Putting this together with Equations 1, 3 and 5 we have that

$$\begin{aligned} \max(r^t_z(e), r^t_z(l)) &< r^t_z(a, b, c, d) \\ &< \min(r^t_z(j, k), r^t_z(f, g)) \end{aligned} \tag{7}$$

We claim that in $R(t)$, for all $t < t_0$,

$$r^t_x(k, l) < r^t_x(e, f) \tag{8}$$

Suppose that this is not true. Then there must exist some $t < t_0$, such that in $R(t)$ either $r^t_z(j, k) < r^t_z(e)$ or $r^t_z(f, g) < r^t_z(l)$. However, by Equation 7 neither of these can hold. So, by contradiction we have that Equation 8 holds for all $t < t_0$.

Now, for $t < t_0$, $\alpha^t_x = r^t_x(a, b)$ and $\beta^t_x = r^t_x(c, d)$. Putting these facts together with Equations 4, 6 and 8, we have that for all $R(t)$ in which $t < t_0$,

$$r^t_x(a, b) < r^t_x(k, l) < r^t_x(e, f) < r^t_x(c, d) \tag{9}$$

By Equations 7 and 9, we conclude that for all $t < t_0$ in $R(t)$ edge (b, c) will intersect both (k, l) and (e, f) with respect to x and z coordinates. Hence, it is not possible that in $R(t_0)$ vertices $b, c, e, \ldots, l$ lie in the same x-z plane. By contradiction we conclude that P and Q do not admit a parallel morph. □

Theorem 2 *There exist parallel orthodisks that do not admit a parallel morph.*

Proof. We construct parallel orthodisks P' and Q' whose boundaries are the drawings P and Q of Figure 1, respectively. By Lemma 1, P and Q do not admit a parallel morph. Therefore, the orthodisks P' and Q' will not admit a parallel morph.

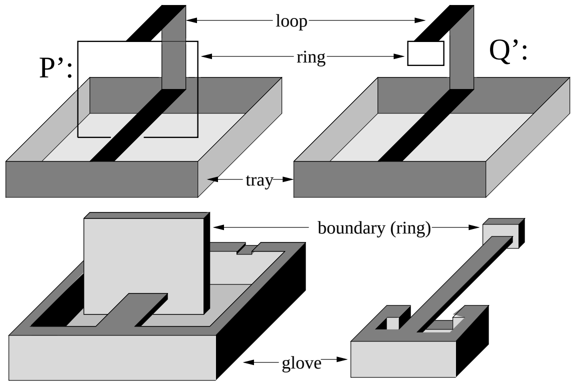

Figure 2: Parallel orthodisks that do not admit a parallel morph.

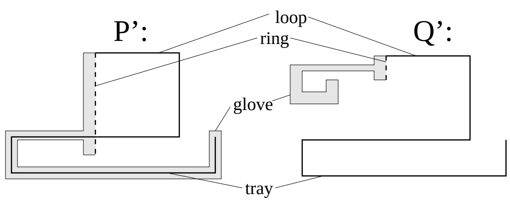

Figure 3: Cross sections of orthodisks P' and Q'.

We begin by adding new vertices, edges and faces to each of P and Q, as illustrated by the topmost two drawings in Figure 2. In particular, to both P and Q we add a lower structure that looks like a box without a top, called the *tray*. Attached to the tray is the *loop*, which consists of three rectangles. The loop connects the tray to a the *ring*, which is simply the boundary of a rectangle. It should be clear that these parallel structures will not admit a parallel morph. However, due to the presence of the ring, the structures are not orthodisks.

To convert these unmorphable structures to orthodisks, we incorporate new parallel orthodisks called *gloves*. To aid in visualizing our construction, imagine that in Q' the glove is a rubber sheet whose boundary coincides with the ring. To get from Q' to P', pass the tray through the ring, extending the rubber surface around the tray. In P' the glove encloses the tray, while in Q' the tray is not enclosed by the glove (see Figure 3).

The lower-most drawings in Figure 2 depict the glove for each of P' and Q'. The boundary of each glove is a rectangle. In both P' and Q' the boundary of the glove is arranged to coincide with the ring. With the addition of the glove, the construction is complete. Note: The edge graph of the gloves is not connected; it is easy to make it connected by adding a single edge. □

4 Open Problems

Many interesting open problems remain, including: What is the complexity of determining whether or not parallel orthogonal genus-0 polyhedra will admit a parallel morph? Are there general conditions under which such polyhedra will always admit a parallel morph? What is the complexity of deciding whether or not two parallel orthogonal drawings of a cycle admit a parallel morph?

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