Drawing planar bipartite graphs with small area

Therese Biedl* Franz Brandenburg†

Abstract

In this paper, we study planar straight-line drawings of bipartite planar graphs. We show that these graphs admit drawings in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid, and that this is optimal.

1 Introduction

A *planar straight-line drawing* of a graph $G = (V, E)$ is an assignment of 2D points to the vertices of G such that if we draw every edge as a straight-line segment between its endpoints, then there are no crossings in the drawing. Such a drawing can exist only if G is *planar*, i.e., can be drawn without crossings in 2D.

Any planar graph has a planar straight-line drawing [6, 13, 14]. Much effort has been put into finding drawings that additionally have a small area in the following sense: Assume that the 2D points assigned to vertices are grid points, i.e., have integer coordinates. We then measure the *width*, *height*, and *area* of the drawing as the width, height, or area of the smallest axis-parallel rectangle that encloses the drawing. Finding the least area bound of grid drawings of planar graphs is listed as a challenging open problem in graph drawing [4]. De Fraysseix, Pach and Pollack showed that every planar graph has a planar straight-line drawing in a $(2n-4) \times (n-2)$-grid [8, 9]. They also gave an example of a planar graph that requires a $(\frac{2}{3}n - 1) \times (\frac{2}{3}n - 1)$-grid in any planar straight-line drawing.

Smaller area was achieved (independently) by Schnyder [12], who showed how to draw a planar graph in an $(n-1) \times (n-1)$-grid. Zhang and He improved this to $(n - \Delta - 1) \times (n - \Delta - 1)$, where $\Delta \geq 0$ is a value derived from the cycle structure of G [15]. Also, every planar graph has a drawing in a $(\frac{2}{3}n - 1) \times 4(\frac{2}{3}n - 1)$-grid [5]; these drawings are optimal in width, but their total area is large.

*School of Computer Science, University of Waterloo, Waterloo, ON N2L 3G1, Canada. biedl@uwaterloo.ca. Research supported by NSERC. Research done while the author was on sabbatical leave at the University of Passau.

†Lehrstuhl für Informatik, Universität Passau, 94030 Passau, Germany. brandenb@fmi.uni-passau.de.

For some special graph classes, further improvements have been made. A graph is called k-*connected* if it remains connected after removing any $k-1$ vertices. Miura, Nishizeki and Nakano showed that all 4-connected planar graphs can be drawn in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid [11]. This does not contradict the lower bound given earlier, since that lower bound graph is not 4-connected. Miura et al. also showed that their bound is tight by exhibiting a 4-connected planar graph that needs an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid in any planar straight-line grid drawing [11].

In this paper, we study another special graph class: planar bipartite graphs. A graph is called *bipartite* if its vertices can be coloured in black and white such that any edge connects a black vertex with a white vertex. In particular, this implies that a bipartite graph has no *triangle*, i.e., a cycle of three vertices; this will be important later on.

Some papers have dealt with how to draw planar bipartite graphs [3, 7], but the emphasis has been on how to display the partition efficiently. In contrast to this, we consider small area as the main focus. We show that any planar bipartite graph can be drawn in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid. We also prove that this is optimal, by exhibiting planar bipartite graphs that require an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid in any planar drawing.

2 The lower bound

We first show that there exists a planar bipartite graph (shown in Figure 1) that requires an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid in any planar straight-line drawing. (In fact, the lower bound holds even if we allow edges to be composed of poly-lines, i.e., to be drawn with bends.) The graph is quite similar to the lower bound example of Miura et al. [11], and consists of $\frac{n-2}{4}$ quadrangles placed around an edge e and connected suitably to obtain a planar bipartite graph.

To draw the edge e, we need either a 0×1-grid or a 1×0-grid; we assume the former. Adding a quadrangle adds at least two units of height and width each to the minimum enclosing box, since we must go "around" the extreme points of the inner

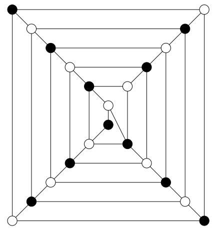

Figure 1: A planar bipartite graph that requires an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid in any planar drawing.

drawing. So in total we must have a width of at least $\frac{n}{2} - 1$ and a height of at least $\frac{n}{2}$, or vice versa, and the lower bound follows.

3 The upper bound

Now we turn to our main result, which is how to draw planar bipartite graphs with small area. We need a few definitions first.

Let G be a planar graph. We assume in the following that some *planar embedding* is fixed, i.e., we are given the circular order of edges around each vertex as it is in some planar drawing of G. This planar embedding defines uniquely the *faces*, i.e., the maximal connected regions of the drawing of G. A *triangle* of a graph is a 3-cycle, i.e., three vertices u, v, w such that edges (u, v), (v, w) and (w, u) exist. Given a planar graph with a fixed planar embedding, a *separating triangle* is a triangle that is not a face; in particular this triangle has vertices both on the inside and the outside in any planar drawing of G that reflects the planar embedding.

The basic approach to draw a planar bipartite graph G is very simple: combine two suitable results. Namely, since G is bipartite, it has no triangle, so in particular no separating triangle. By a result of the first author, Kant and Kaufmann [2], a planar graph without separating triangle can be made 4-connected and planar by adding edges. Now combine this with the result of Miura et al. [11] for 4-connected planar graphs, and any planar bipartite graph can be drawn in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid.

Unfortunately, the proof isn't quite this easy, because we have not spoken the complete truth in citing the results above. There are two problems:

- There is one exception to the result of Biedl et al. [2]: So-called *graphs containing a star* (defined below) can be drawn without separating triangle, but cannot be made 4-connected by adding edges. We thus must argue why planar bipartite graphs (with one exception) do not contain a star.
- The algorithm by Miura at al. [11] works for planar 4-connected graphs for which one face has at least 4 vertices. On the other hand, we make graphs 4-connected by turning all faces into triangles. We thus must extend the result of Miura at al. [11] to what we call *almost 4-connected planar graph*, and show that any planar bipartite planar graph can be made almost 4-connected.

3.1 Graphs containing stars

From now on, let G be a planar graph with a fixed planar embedding. We say that G *contains a star with central vertex w at face F* if every vertex on F is either w or adjacent to w, and F contains at least four distinct vertices that are not w. In particular, the *star graph* (a tree where one vertex is adjacent to all others) contains a star. See Figure 2.

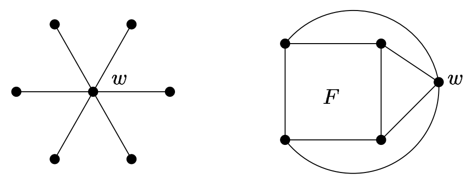

Figure 2: The star graph, and some other graph containing a star with central vertex w at face F.

A graph is called *triangulated* if every face of it is a triangle. In the right graph in Figure 2, we cannot add any edge and stay planar without creating a separating triangle. Thus, this graph (and in fact, any graph that contains a star) cannot be triangulated by adding edges without introducing a separating triangle. On the other hand, graphs containing a star are the only 2-connected graphs where this is not possible.

Theorem 1 *[2] Let G be a 2-connected planar graph with a fixed embedding. Then G can be triangulated without adding separating triangles if and only if G does not contain a star.*

Now let G be a planar bipartite graph. If G is not the star graph, then we can add edges to G to make it *maximal planar bipartite*, i.e., every face of G has exactly four vertices on it. It is known that a maximal planar bipartite graph is 2-connected.

Claim 1 *A 2-connected planar bipartite graph G does not contain a star.*

Proof. Assume G contained a star at face F with central vertex w, The boundary of F is a simple cycle (by 2-connectivity) and hence contains w at most once. So there are at least two consecutive vertices on F that are not w, and hence adjacent to w. This gives a triangle; a contradiction. □

So for any planar bipartite graph G except the star graph, we can add edges to G until it is maximal planar bipartite, and then add further edges until it is triangulated without separating triangles. Such a graph is known to be 4-connected. We hence obtain the following result:

Corollary 2 *Any planar bipartite graph except the star graph can be made 4-connected and planar by adding edges.*

3.2 Drawing 4-connected graphs

The algorithm by Miura et al. [11] to draw a 4-connected planar graph G relies crucially on the so-called *4-canonical ordering* of G, defined as follows. For an ordering $v_1, \ldots, v_n$ of the vertices, let G_k be the subgraph induced by $v_1, \ldots, v_k$, and let $\overline{G_k}$ be the graph induced by the vertices $v_{k+1}, \ldots, v_n$. A vertex ordering is called a 4-canonical ordering if

- (v_1, v_2) and (v_{n-1}, v_n) are edges on the outer-face. In particular, the outer-face must have at least four vertices.
- For each k with $2 \leq k \leq n-2$, v_k is on the outer-face of G_k, has at least 2 neighbours in G_k, and at least 2 neighbours in $\overline{G_{k-1}}$.

Such an ordering exists for 4-connected planar graph for which the outer-face has four vertices and which is *internally triangulated* (any interior face is a triangle) [10]. Now we extend this to more graphs. We say that a planar graph G is *almost 4-connected* if it is internally triangulated, the outer-face contains 4 vertices, and G becomes 4-connected if we add one edge on the outer-face. In particular, we get an almost 4-connected graph if we make a graph 4-connected by adding edges without adding separating triangles, but omit one added edge, and make the face that contained it the outer-face. Therefore, any planar bipartite graph can be made almost 4-connected. Now we need:

Lemma 3 *Any almost 4-connected planar graph has a 4-canonical ordering.*

Proof. To see this, we need to inspect the proof of the existence of a 4-canonical ordering by Kant and He [10]. This proof proceeds by reverse induction, i.e., it first defines v_n, then v_{n-1}. All that is required is that we pick adjacent vertices v_n and v_{n-1} such that G_{n-2} (i.e., the graph left over after v_n and v_{n-1} are deleted) is 2-connected. We now show that we can pick such v_n and v_{n-1} even for almost 4-connected graphs.

Let G be almost 4-connected, and let e be an edge such that $G \cup e$ is 4-connected. Let v_n be one endpoint of e, and let v_{n-1} be one of the neighbours of v_n on the outer-face. Since $G \cup e$ is 4-connected, $G \cup e - v_n$ is 3-connected. But since e is adjacent to v_n, $G \cup e - v_n$ is the same as $G - v_n$, so $G - v_n$ is 3-connected, and $G - \{v_n, v_{n-1}\}$ is 2-connected as desired. □

Putting everything together, we arrive at the main result:

Theorem 4 *Every planar bipartite graph G has a planar straight-line grid drawing in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid. Such a drawing can be found in linear time.*

Proof. If G is the star graph, then such a drawing can be found easily. Otherwise, embed G arbitrarily and add edges to it to make it maximal planar bipartite. Now the graph is 2-connected and bipartite; it hence contains no star by Claim 1 and no separating triangle by bipartiteness. Add edges to the graph to make it triangulated without adding separating triangles [2]; now the graph is 4-connected. We must have added at least one edge to the outer-face since it is now a triangle. Delete this added edge; the resulting graph G' is a super-graph of G and almost 4-connected. Compute a 4-canonical ordering of G' ([10] and Lemma 3) and use it to draw G' in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid [11]. Delete all added edges; this is the desired drawing of G. Since all required ingredients can be implemented in linear time [2, 11, 10], the algorithm runs in linear time. □

4 Triangle-free planar graphs

On closer inspection, the reader should notice that only at very few places did we actually use that the input graph is bipartite. In fact, we can achieve an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid for any planar graph that can be made almost 4-connected by adding edges. By Theorem 1, this in turn can be achieved for any planar graph that is 2-connected, does not contain a star and is not triangulated. So we now explore another graph class for which this is the case.

Consider *triangle-free planar graphs*, i.e., planar graphs that do not contain a triangle. Clearly they do not contain a separating triangle in any planar embedding. The proof of Claim 1 only uses that bipartite graphs have no triangles, so this claim also holds for triangle-free graphs.

All that remains to do is to show that triangle-free planar graphs can be made 2-connected by adding edges, without adding triangles. (For planar bipartite graphs, we did this by making them maximal planar bipartite.) The "normal" approach to making a planar graph biconnected ("add an edge between two consecutive neighbours of a cut-vertex that are in different biconnected components") does not work here, since this may introduce a triangle. However, by modifying the approach, this can be achieved.

Lemma 5 *Any triangle-free planar graph G except the star graph can be made 2-connected planar by adding edges without adding triangles.*

Proof. (Sketch) If G is a tree, then it is bipartite and the claim holds by making it maximal planar bipartite. If G is not a tree, then it has some maximal 2-connected component C that contains more than one edge, and for which every face hence is a simple cycle. Let v be a cut-vertex in C, and let C_1 be some other 2-connected component incident to C. We connect a neighbour of v in C_1 with a vertex in C that shares a face with v, but is not adjacent to v; this exists since G has no triangle and every face in C hence is a cycle of length at least 4. □

Now, as before, apply Theorem 1 to make a 2-connected triangle-free graph into a 4-connected graph, omit the last edge to make it almost 4-connected, and then draw the resulting graph. We thus obtain:

Theorem 6 *Any triangle-free planar graph can be drawn in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid.*

5 Conclusion

In this paper, we showed that any planar bipartite graph (and in fact, any triangle-free planar graph) has a planar straight-line drawing in an $\lfloor \frac{n}{2} \rfloor \times (\lceil \frac{n}{2} \rceil - 1)$-grid, and this is optimal since some planar bipartite graphs require this grid size.

Our main open question is whether there is an easier proof for our result. Do we really need the heavy-duty machinery of making the graph 4-connected first? We recently showed that maximum planar bipartite graphs have a special vertex ordering of their own (where every white vertex has exactly one incoming edge, whereas every black vertex has at least two incoming edges) [1]. Can we obtain small drawings of planar bipartite graphs directly, using this vertex ordering?

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