On some monotone path problems in line arrangements

Adrian Dumitrescu*

Abstract

We give tight estimates on the minimum length of a longest monotone path in an arrangement of n lines, where length counts the number of turns on the path. Tight estimates are also obtained for the case when length counts the number of visited vertices. When length is defined as the size of a convex/concave chain in the arrangement an exact bound is obtained.

1 Introduction

Consider a set $\mathcal{L}$ of n lines in the plane. The lines of $\mathcal{L}$ induce a cell complex, $\mathcal{A}(\mathcal{L})$, called the *arrangement* of $\mathcal{L}$, whose *vertices* are the intersection points of the lines, whose *edges* are the maximal portions of lines containing no vertices and whose two-dimensional *cells* are the connected components of $\mathbb{R}^2 \setminus \cup_{\ell \in \mathcal{L}} \ell$. It is assumed that none of the lines is parallel to the y-axis.

One of the properties of a line arrangement with n lines is the maximum possible length, denoted by λ_n, of an x-monotone polygonal line (path) composed of edges of the arrangement. The length is defined as the number of turns of the polygonal line plus one (i.e., the number of segments of the polygonal path). The problem to estimate λ_n was posed in [4]. The best known lower bound, due to Balogh et. al. [1], is subquadratic: $\Omega(n^2/C^{\sqrt{\log n}})$, where $C > 1$. It improved on earlier results due to Sharir [3, 4], Matoušek [6], Radoičić and Tóth [7]. From the opposite direction, no subquadratic upper bound is known. Such bounds have been recently obtained only for line arrangements with a small number of slopes [2].

In this paper we consider some related questions on monotone paths. An arrangement of lines, $\mathcal{A}(\mathcal{L})$, is called *simple* if no two lines in $\mathcal{L}$ are parallel and no three lines pass through the same point (vertex). For the problems that we are considering, we will further assume the arrangement is simple. Let $\mathcal{A}$ be an arrangement of n lines; we write $|\mathcal{A}| = n$. We denote by $V(\mathcal{A})$ the set of its vertices. By the above assumption, the total number of vertices in the arrangement is $|V(\mathcal{A})| = \binom{n}{2}$.

The k-level of an arrangement of n lines is the closure of the set of points on the lines with the property that there are exactly k lines below them ($k = 0, \ldots, n-1$). The k-level of a line arrangement is also a x-monotone polygonal path, which turns at each vertex of the arrangement that lies on the path. Let $l(\mathcal{A})$ be the length of a longest level in $\mathcal{A}$, where length is the number of vertices plus one, i.e., the number of segments on the level. Put $l(n) = \min_{|\mathcal{A}|=n} l(\mathcal{A})$.

Let $t(\mathcal{A})$ be the length of a longest monotone path in $\mathcal{A}$, where length is the number of turns plus one, i.e., the number of segments on the path. Put $t(n) = \min_{|\mathcal{A}|=n} t(\mathcal{A})$. Clearly $l(n) \le t(n)$.

Theorem 1 *Each simple arrangement of n lines admits a monotone path of length at least n, where length is the number of turns plus one. This bound is asymptotically tight: for each $n \ge 2$, there exists a line arrangement in which no monotone path is longer than $4n/3 + \log n$. Thus $n \le l(n) \le t(n) \le \frac{4n}{3}(1 + o(1))$.*

Radoičić and Tóth have observed that if length is defined as the number of vertices of the arrangement visited by a monotone path, it is easy to construct examples which admit paths of length $\Omega(n^2)$. We show that this can be further strengthened.

Theorem 2 *For each $n \ge 2$ there exists a simple arrangement of n lines that admits a monotone path which visits all its vertices.*

Let $v(\mathcal{A})$ be the maximum number of vertices visited by a monotone path in $\mathcal{A}$ plus one. Put $v(n) = \min_{|\mathcal{A}|=n} v(\mathcal{A})$.

Theorem 3 *Each simple arrangement of n lines admits a monotone path which visits at least $n-1$ vertices. This bound is asymptotically tight: for each $n \ge 2$, there exists a line arrangement in which no monotone path visits more than $3n/2 + \log n$ vertices. Thus $n \le v(n) \le \frac{3n}{2}(1 + o(1))$.*

Let $c_1(\mathcal{A})$ (resp. $c_2(\mathcal{A})$) be the length of a longest monotone convex (resp. concave) chain of $\mathcal{A}$, where length is the number of turns plus one (i.e., the number of segments in the chain), and write $c(\mathcal{A}) = \max(c_1(\mathcal{A}), c_2(\mathcal{A}))$, $c(n) = \min_{|\mathcal{A}|=n} c(\mathcal{A})$. We have $c(n) \le t(n) \le v(n)$.

Theorem 4 *Each simple arrangement of n lines admits either a monotone convex chain or a monotone concave chain of length at least $\frac{\log n}{2}(1 + o(1))$. This bound is tight. More precisely, let $N = N(n)$ be the minimum number such that any simple arrangement of N lines admits a convex or concave chain of length n. Then $N(n) = \binom{2n-4}{n-2} + 1$. We thus have $c(n) = \frac{\log n}{2}(1 + o(1))$.*

A set of points in the plane is in *general position* if no three points are collinear. A finite set of points is in *convex*

*Computer Science, University of Wisconsin–Milwaukee, 3200 N. Cramer Street, Milwaukee, WI 53211, USA; ad@cs.uwm.edu

position if the points are the vertices of a convex polygon. The following classical result of Erdős and Szekeres is well known:

Theorem 5 [5] *For any $n \geq 3$ there exists an integer $f(n)$ such that any set of at least $f(n)$ points in general position in the plane contains n points in convex position.*

The statement regarding the number of lines $N(n)$ in Theorem 4 is a direct consequence of the original proof of Erdős and Szekeres regarding points in convex position.

2 Making turns: proof of Theorem 1

We first show that every n-line arrangement $\mathcal{A}$ admits a monotone path of length at least n. Consider the n levels in the arrangement $L_0, \ldots, L_{n-1}$. It is well known that each level is a monotone path that turns at each vertex it passes through. Every vertex of $\mathcal{A}$ appears exactly in two consecutive levels, L_k and L_{k+1} for some $k \in \{0, \ldots, n-2\}$. This gives

$$|L_0| + \ldots |L_{n-1}| = 2\binom{n}{2} = n(n-1),$$

where $|L_k|$ is the complexity (i.e., number of vertices) of level k. Let i be such that $|L_i| = \max\{|L_1|, \ldots, |L_n|\}$. Then $|L_i| \geq n-1$, and L_i is a monotone path of length at least n. Hence $t(n) \geq l(n) \geq n$.

For two sets of points $P, Q \subset \mathbb{R}^2$, we write $P < Q$ if $x(p) < x(q)$ for any pair of points $p \in P$, $q \in Q$.

We now show the upper bound in Theorem 1. We recursively construct an arrangement of lines $\mathcal{A}$ by putting together two arrangements, one with $\lfloor n/2 \rfloor$ lines, $\mathcal{A}_1$, and one with $\lceil n/2 \rceil$ lines, $\mathcal{A}_2$. See Fig. 1. The lines in each $\mathcal{A}_i$, $i = 1, 2$ are almost parallel to each other, and the slopes of the lines in $\mathcal{A}_i$ are close to m_i, where $m_1 < m_2$. The minimum slope of lines in $\mathcal{A}_i$ is equal to m_i, and the slopes of the lines in $\mathcal{A}_1$ are smaller than the slopes of the lines in $\mathcal{A}_2$. In addition, all the vertices of $\mathcal{A}_1$ and $\mathcal{A}_2$ lie left from those between any pair of lines (ℓ_i, ℓ_j), $\ell_i \in \mathcal{A}_1$ and $\ell_j \in \mathcal{A}_2$. That is, $V(\mathcal{A}_1) \cup V(\mathcal{A}_2) < V(\mathcal{A}) \setminus (V(\mathcal{A}_1) \cup V(\mathcal{A}_2))$. The recursive step requires compressing the resulting arrangements with respect to the lines of slopes m_1 and m_2 respectively before combining them.

Let $t(n, j$ be the maximum length of a monotone path in the arrangement $\mathcal{A}$ with n lines, whose last segment is on line j, $j = 1, \ldots, n$ (we index the lines in increasing order of slope). We claim that for any n,

$$t(n,j) \leq \begin{cases} 2n/3 + 2j + \log n & \text{if } 1 \leq j \leq n/3 \\ 4n/3 + \log n & \text{if } n/3 \leq j \leq 2n/3 \\ 8n/3 - 2j + \log n & \text{if } 2n/3 \leq j \leq n \end{cases}$$

and note that this inequality implies our bound. We proceed by induction on n. The basis is satisfied since $t(1, 1) = 1$ and $t(2, 1) = t(2, 2) = 2$. Let $n \geq 3$. Consider a monotone path p in $\mathcal{A}$. We distinguish four cases as to how p enters and leaves the staircase junction formed by the lines of $\mathcal{A}_1$ and $\mathcal{A}_2$. See Fig. 2. Let i denote the index of the line on which p leaves $\mathcal{A}_1$ or $\mathcal{A}_2$ before entering the junction. Each case has six subcases accounting for which interval i and j belong. To avoid unnecessary details we omit ± 1 terms, floors and ceilings as well as the log terms in verifying each of the cases. The logarithmic term in the bound covers all these.

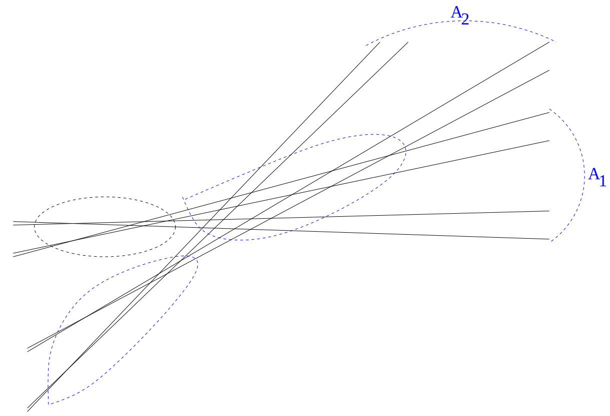

Figure 1: Arrangement of eight lines obtained recursively.

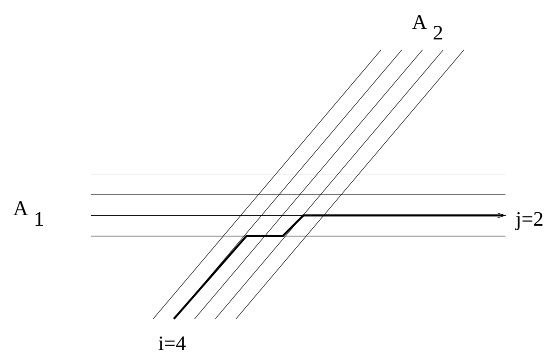

Figure 2: Staircase junction in an arrangement of nine lines, and a monotone path as in case 1 of the proof of Theorem 1.

Case 1: p enters the junction on a line of $\mathcal{A}_2$ and leaves it on a line of $\mathcal{A}_1$. We have $1 \leq i \leq \frac{n}{2}$ and $1 \leq j \leq \frac{n}{2}$. The number of turns in the junction is at most $2\min(i, j) - 1 \leq 2\min(i, j)$.

Subcase 1.1: $1 \leq i \leq \frac{n}{6}$ and $1 \leq j \leq \frac{n}{3}$.

$$t(n,j) \leq \frac{2}{3}\frac{n}{2} + 2i + 2\min(i,j) \leq \frac{2}{3}n + 2j, \text{ for } 2i \leq \frac{n}{3}.$$

Subcase 1.2: $\frac{n}{6} \leq i \leq \frac{n}{3}$ and $1 \leq j \leq \frac{n}{3}$.

$$t(n,j) \leq \frac{4}{3}\frac{n}{2} + 2\min(i,j) \leq \frac{2}{3}n + 2j, \text{ clearly holds.}$$

Subcase 1.3: $\frac{n}{3} \leq i \leq \frac{n}{2}$ and $1 \leq j \leq \frac{n}{3}$.

$$t(n,j) \leq \frac{8}{3}\frac{n}{2} - 2i + 2\min(i,j) \leq \frac{2}{3}n + 2j, \text{for } 2i \geq \frac{2n}{3}.$$

Subcase 1.4: $1 \le i \le \frac{n}{6}$ and $\frac{n}{3} \le j \le \frac{n}{2}$.

$$t(n,j) \le \frac{2}{3}\frac{n}{2} + 2i + 2\min(i,j) \le \frac{4}{3}n, \text{ since } 4i \le n.$$

Subcase 1.5: $\frac{n}{6} \le i \le \frac{n}{3}$ and $\frac{n}{3} \le j \le \frac{n}{2}$.

$$t(n,j) \le \frac{4}{3}\frac{n}{2} + 2\min(i,j) \le \frac{4}{3}n, \text{ since } 2i \le \frac{2n}{3}.$$

Subcase 1.6: $\frac{n}{3} \le i \le \frac{n}{2}$ and $\frac{n}{3} \le j \le \frac{n}{2}$.

$$t(n,j) \le \frac{8}{3}\frac{n}{2} - 2i + 2\min(i,j) \le \frac{4}{3}n, \text{ clearly holds.}$$

The remaining three cases are omitted for lack of space.

3 Visiting vertices: proofs of Theorem 2 and Theorem 3

We start with the proof of Theorem 2. The arrangement in Fig. 3 admits a monotone path which visits all its vertices. Consider first the case of even n. The arrangement can be

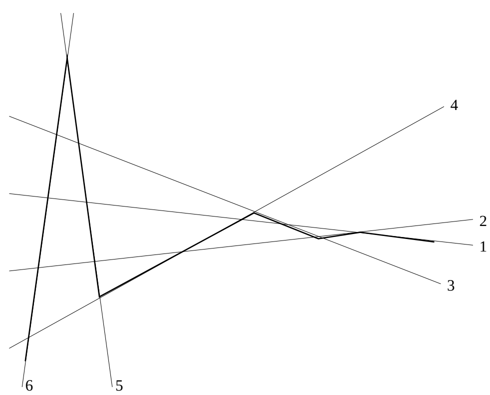

Figure 3: Arrangement of six lines and a monotone path (in bold) which visits all 15 vertices of the arrangement.

iteratively constructed by adding the lines in pairs. First add two lines making a small angle with the horizontal axis. Denote by $\mathcal{A}_{2i}$ the arrangement formed by the first $2i$ lines for $i = 1, 2, \ldots, n/2$. Having placed the first $2(i-1)$ lines, add two lines ℓ_{2i-1} and ℓ_{2i} to get $\mathcal{A}_{2i}$, where ℓ_{2i} has a sufficiently large positive slope and ℓ_{2i-1} has a sufficiently large negative slope. In addition, (*i*) the point of intersection between ℓ_{2i-1} and ℓ_{2i} lies above any of the previous lines and (*ii*) the points of intersection between ℓ_{2i-1} and the previous lines and those between ℓ_{2i} and and the previous lines lie left of all vertices of $\mathcal{A}_{2i-2}$, i.e., $V(\mathcal{A}_{2i}) \setminus V(\mathcal{A}_{2i-2}) < V(\mathcal{A}_{2i-2})$. It is clear that the construction can be iterated as many times as desired. The monotone path which follows lines $\ell_n, \ldots, \ell_1$ in this order — illustrated in Fig. 3 for $n = 6$ — visits all vertices. For the case of odd n, line ℓ_{2i} is removed along with the portion of the monotone path which it supports.

We now prove Theorem 3. Since $v(n) \ge t(n)$, the lower bound follows from Theorem 1. We now show the upper bound in Theorem 3. Let $v(n, j$ be the maximum length of a monotone path in the arrangement $\mathcal{A}$ with n lines, whose last segment is on line j, where $j = 1, \ldots, n$ (we index the lines in increasing order of slope). We claim that for any n,

$$v(n,j) \le n + \min(j, n-j) + \log n,$$

and note that this inequality implies our bound. We proceed by induction on n. The basis, $n = 1$ and $n = 2$ is satisfied. Let $n \ge 3$. Consider a monotone path p in $\mathcal{A}$. We distinguish four cases as to how p enters and leaves the staircase junction formed by the lines of $\mathcal{A}_1$ and $\mathcal{A}_2$. See Fig. 2. Let i denote the index of the line on which p leaves $\mathcal{A}_1$ or $\mathcal{A}_2$ before entering the junction.

Case 1: p enters the junction on a line of $\mathcal{A}_2$ and leaves it on a line of $\mathcal{A}_1$. We have $1 \le i \le \lceil n/2 \rceil$. The induction hypothesis gives $v(n,j) \le \lceil n/2 \rceil + \min(i, \lceil n/2 \rceil - i) + \log \lceil n/2 \rceil + (i + j - 1)$, where the term $i + j - 1$ bounds the number of vertices visited in the junction. Replacing $\min(i, \lceil n/2 \rceil - i)$ by $\lceil n/2 \rceil - i$, it is enough to verify that $2\lceil n/2 \rceil - i + i + j - 1 + \log \lceil n/2 \rceil \le n + j + \log n$. The inequality clearly holds.

Case 2: p enters the junction on a line of $\mathcal{A}_2$ and leaves it on a line of $\mathcal{A}_2$. We have $1 \le i \le \lceil n/2 \rceil$ and $j \ge \lfloor n/2 \rfloor + 1$. The induction hypothesis gives $v(n,j) \le \lceil n/2 \rceil + \min(i, \lceil n/2 \rceil - i) + \log \lceil n/2 \rceil + (2\lfloor n/2 \rfloor + i - j)$, where the term $2\lfloor n/2 \rfloor + i - j$ bounds the number of vertices visited in the junction. Replacing $\min(i, \lceil n/2 \rceil - i)$ by $\lceil n/2 \rceil - i$, we aim to show $2\lceil n/2 \rceil - i + 2\lfloor n/2 \rfloor + i - j + \log \lceil n/2 \rceil \le 2n - j + \log n$. The inequality obviously holds since $\log \lceil n/2 \rceil \le \log n$.

Case 3: p enters the junction on a line of $\mathcal{A}_1$ and leaves it on a line of $\mathcal{A}_1$. We have $1 \le i \le j \le \lfloor n/2 \rfloor$. The induction hypothesis gives $v(n,j) \le \lfloor n/2 \rfloor + \min(i, \lfloor n/2 \rfloor - i) + \log \lfloor n/2 \rfloor + \lceil n/2 \rceil + j - i$, since the number of vertices visited in the junction is not more than $\lceil n/2 \rceil + j - i$. Replacing $\min(i, \lfloor n/2 \rfloor - i)$ by i, we aim to show that $\lfloor n/2 \rfloor + i + \lceil n/2 \rceil + j - i + \log \lfloor n/2 \rfloor \le n + j + \log n$. The inequality is obvious.

Case 4: p enters the junction on a line of $\mathcal{A}_1$ and leaves it on a line of $\mathcal{A}_2$. We have $1 \le i \le \lfloor n/2 \rfloor$ and $j \ge \lfloor n/2 \rfloor + 1$. The induction hypothesis gives $v(n,j) \le \lfloor n/2 \rfloor + \min(i, \lfloor n/2 \rfloor - i) + \log \lfloor n/2 \rfloor + (n + 1 - i - j + \lfloor n/2 \rfloor)$. The last term bounds the number vertices visited in the junction. Replacing $\min(i, \lfloor n/2 \rfloor - i)$ by i, we want to verify that $\lfloor n/2 \rfloor + i + n + 1 - i - j + \lfloor n/2 \rfloor + \log \lfloor n/2 \rfloor \le 2n - j + \log n$. The inequality follows from: $\log \lfloor n/2 \rfloor + 1 \le \log n$.

4 Making left (or right) turns only: convex/concave chains and the proof of Theorem 4

A set X of points in general position in the plane, no two on a vertical line, is an *n-cap* (*n-cup*, respectively) if X is in convex position and all points of X lie above (below, respectively) the line connecting the leftmost point of X with the rightmost point of X (see Fig. 4).

Figure 4: A 4-cap and a 5-cup.

Erdős and Szekeres proved that any set of at least $\binom{2n-4}{n-2}+1$ points in general position in the plane, no two on a vertical line, contains an n-cup or an n-cap [5]. They showed that this bound is tight, i.e., there exist sets with $\binom{2n-4}{n-2}$ points containing no n-cup or n-cap. More generally, there exist sets with $\binom{k+l-4}{k-2}$ points containing no k-cup or l-cap.

Proving the main part of Theorem 4 is an exercise in using the point-line duality transform. Fix an (x, y)-coordinate system in the plane and consider the duality transform D which maps a point $p = (a, b)$ to the nonvertical line p^* with equation $y = ax - b$. Conversely, a nonvertical line l with equation $y = ax + b$ is mapped to the point $l^* = (a, -b)$.

Lemma 6 *A set of $n \geq 3$ points forms an n-cup (n-cap, respectively) if and only if the dual lines form a monotone convex (concave, respectively) path of length n.*

Theorem 5 implies via Lemma 6 that $N(n) \leq \binom{2n-4}{n-2}+1$. As mentioned above, there exist sets with $\binom{2n-4}{n-2}$ points in general position, no two on a vertical line containing no n-cup or n-cap. The dual set of lines forms a simple arrangement with $\binom{2n-4}{n-2}$ lines that has no monotone convex or concave path of length n, thus $N(n) = \binom{2n-4}{n-2} + 1$. The asymptotic rate of the above binomial coefficient gives $c(n) = \frac{\log n}{2}(1 + o(1))$, concluding the proof of Theorem 4.

5 Conclusion

Our constants in the upper bounds in Theorem 1 and Theorem 3 are best possible for the arrangement in Fig. 1. Consider first the number of turns. By repeatedly using $i \approx n/3$ (on a line of $\mathcal{A}_2$) and $j \approx n/3$ at each step of the recursion, one gets a monotone path of length $\approx \frac{2}{3}n + \frac{2}{3}\frac{n}{2} + \ldots = \frac{2n}{3}\sum_{i=0}^{\infty}\frac{1}{2^i} = \frac{4n}{3}$.

Consider now the number of visited vertices. By repeatedly using $i \approx n/4$ (on a line of $\mathcal{A}_2$) and $j \approx n/2$ at each step of the recursion, one gets a monotone path of length $\approx \frac{3}{4}n + \frac{3}{4}\frac{n}{2} + \ldots = \frac{3n}{4}\sum_{i=0}^{\infty}\frac{1}{2^i} = \frac{3n}{2}$.

Clearly, taking a walk on any line of an arrangement visits $n-1$ vertices (which gives another proof of the trivial bound $v(n) \geq n$). Are there monotone paths (in any arrangement) which visit $(1+\delta)n$ vertices for some constant $\delta > 0$?

For a set S of n points in the plane, a subset S' of S is called a *k-set of S*, $1 \leq k \leq n-1$, if S' has exactly k points and it can be cut off S by a straight line disjoint from S (see e.g. [8]). It is straightforward to construct examples of point sets whose number of k-sets is n for each $k \in \{1, \ldots, n-1\}$, namely points in convex position. The point-line duality transform provides examples of line arrangements where the complexity of each level is roughly at most $2n$. It is not clear however whether this bound can be brought down to about n: Does $v(n) = n(1 + o(1))$ hold? If not, does any of $t(n) = n(1 + o(1))$ or $l(n) = n(1 + o(1))$ hold?

References

[1] J. Balogh, O. Regev, C. Smyth, W. Steiger and M. Szegedy, Long monotone paths in line arrangements, *Nineteenth Annual ACM Symposium on Computational Geometry*, June 2003, 124–128.

[2] A. Dumitrescu, Monotone paths in line arrangements with a small number of directions, *Discrete & Computational Geometry*, accepted.

[3] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer-Verlag, Heidelberg, 1987.

[4] H. Edelsbrunner and L. Guibas, Topologically sweeping an arrangement, *Journal of Computer and System Sciences*, **38** (1989), 165–194.

[5] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica*, **2** (1935), 463–470.

[6] J. J. Matoušek, Lower bounds on the length of monotone paths in arrangements, *Discrete & Computational Geometry*, **6** (1991), 129–134.

[7] R. Radoičić and G. Tóth, Monotone paths in line arrangements, *Proceedings of the Seventeenth Annual ACM Symposium on Computational Geometry*, 2001, 312–314.

[8] E. Welzl, More on k-sets of finite sets in the plane, *Discrete & Computational Geometry*, **1** (1986), 95–100.