A remark on the Erdős–Szekeres theorem

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Abstract

The Erdős–Szekeres theorem states that for any $n \geq 3$, any sufficiently large set of points in general position in the plane contains n points in convex position. We show that for any $k \geq 1$ and any finite sequence $h_0, h_1, \ldots, h_k$, with $h_i \geq 3$ ($i = 0, \ldots, k$), any sufficiently large set of points in general position in the plane contains either an empty convex h_0-gon or k convex polygons $P_1, \ldots, P_k$, where $|P_i| = h_i$ and for each $i \in \{1, \ldots, k-1\}$, P_i strictly contains P_{i+1} in its interior.

1 Introduction

A set of points in the plane is in *general position* if no three points are collinear. A finite set of points is in *convex position* if the points are the vertices of a convex polygon. The following classical result of Erdős and Szekeres is well known:

Theorem 1 [3] *For any $n \geq 3$ there exists an integer $f(n)$ such that any set of at least $f(n)$ points in general position in the plane contains n points in convex position.*

We offer the following generalization:

Theorem 2 *For any $k \geq 1$ and any finite sequence $h_0, h_1, \ldots, h_k$, with $h_i \geq 3$ ($i = 0, \ldots, k$), there is an $N = N(h_0, h_1, \ldots, h_k)$ such that any set of at least N points in general position in the plane contains either*

(i) *an empty convex h_0-gon, or*

(ii) *k convex polygons $P_1, \ldots, P_k$, where $|P_i| = h_i$, and such that for each $i \in \{1, \ldots, k-1\}$, P_i strictly contains P_{i+1} in its interior.*

The special case $k = 1$ and $h_0 = h_1 = n$ is clearly equivalent to Theorem 1.

In 1975, Erdős [4] asked whether the following sharpening of the Erdős–Szekeres theorem holds: Is there a smallest number $g(n)$ such that any set of at least $g(n)$ points in general position in the plane contains an empty convex n-gon? This was answered in the negative by Horton [5] who constructed arbitrary large point sets without any empty convex heptagon (thus $g(n)$ does not exist for $n \geq 7$). The existence of $g(n)$ is known for small n, namely for $n = 3, 4, 5$ and is still a mystery for $n = 6$.

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Let S be a set of points in general position in the plane. S is called *k-convex* if each triangle determined by S contains at most k points in its interior. Recently, Valtr [9] showed that k-convexity is sufficient to guarantee the existence of a large empty convex polygon if the point set is large enough.

Theorem 3 [9] *For any $k \geq 0$ and $n \geq 3$ there exists an integer $g(k, n)$ such that any k-convex set of at least $g(k, n)$ points in general position in the plane contains an empty convex n-gon.*

The Erdős–Szekeres theorem has inspired quite a few generalizations along the years, some of which are still unsolved. We direct the reader to the survey [6] and the references therein. In particular, Bialostocki et al. [1] have proposed a conjecture which is now still only partially solved, while Valtr [9] has recently conjectured a surprising sharpening of the old Erdős–Szekeres result. Various extensions of the Erdős–Szekeres result for families of convex sets appear in [2, 7, 8].

2 Proof of Theorem 2

Note that the statement is trivial for $h_0 = 3$. We proceed by induction on k. The basis $k = 1$ follows from Theorem 1. Assume now that $k \geq 2$. Let $h' = \max(h_0, h_1)$ and

$$N = f(h' \cdot g(N(h_0, h_2, \ldots, h_k) - 1, h_0)),$$

where f and g are those in Theorem 1 and Theorem 3, and the existence of $N(h_0, h_2, \ldots, h_k)$ is guaranteed by induction. Let S be a set of at least N points in general position in the plane. By Theorem 1, it contains a convex polygon P with $h' \cdot g(N(h_0, h_2, \ldots, h_k) - 1, h_0)$ vertices. By drawing a suitable set of non-crossing diagonals, P can be divided into at least $g(N(h_0, h_2, \ldots, h_k) - 1, h_0)$ convex h'-gons with disjoint interiors. If at least one of them is empty in S, Theorem 2 follows (since $h' \geq h_0$).

Assume therefore that each such h'-gon contains at least one point, thus P contains at least $g(N(h_0, h_2, \ldots, h_k) - 1, h_0)$ points of S in its interior. Denote by X this set of points. Either each triangle determined by X contains at most $N(h_0, h_2, \ldots, h_k) - 1$ points inside, or there exists a triangle with at least $N(h_0, h_2, \ldots, h_k)$ points inside. In the first case, by Theorem 3, X contains an empty convex polygon with h_0 vertices, which is also empty in S. In the second case, let Δabc be a triangle determined by X with at least $N(h_0, h_2, \ldots, h_k)$ points inside. By induction, these points either determine an empty convex h_0-gon, which is also empty in S, or they determine a sequence of convex polygons $P_2, \ldots, P_k$, where $|P_i| = h_i$ for each $i \in \{2, \ldots, k\}$, and such that for each $i \in \{2, \ldots, k-1\}$, P_i strictly contains P_{i+1} in its interior. Recall that Δabc lies in the interior of P. Since $|P| \geq h_1$, one can add points of P one by one to $\{a, b, c\}$ while updating the convex hull of the resulting set, until there are exactly h_1 points on the hull boundary (as shown below). One obtains in this way the configuration of points in part (*ii*) of the theorem.

Indeed, the size of the convex hull of the set of points obtained by repeatedly adding one point of P, is three to start with, and $|P|$ in the end (if all points of P where to be added). At each point addition, the size of the convex hull can increase by at most one (and could decrease by as much as two). Therefore there exists one step at which the size of the convex hull is exactly h_1.

References

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