Computing Nice Sweeps for Polyhedra and Polygons

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1 Introduction

The plane sweep technique is one of the best known paradigms for the design of geometric algorithms [2]. Here an imaginary line sweeps over the plane while computing the property of interest at the moment the sweep line passes the required information needed to compute that property. There are also three-dimensional problems that are solved by space sweep, where a plane sweeps the space.

This paper does not deal with the sweeping paradigm itself; it deals with testing polygons and polyhedra to determine if they have a certain property. The properties that we consider are related to sweeping. We will test for a simple polygon or polyhedron if it can be swept by a line or plane such that every cross-section has a property like being convex or simply-connected. For example, to determine for a simple polygon (with interior) in the plane whether there is a sweep direction such that every cross-section is simply-connected (a point, line segment, or empty) is the well-known question of determining whether a simple polygon is monotone in some direction. We solve two extensions of this problem in 3-space, and solve another extension in the plane.

The first question we address applies to a polyhedron P in 3-space. We want to determine if there is a vector $\vec{d}$, such that if a sweeping plane with normal $\vec{d}$ passes over P, every cross-section of P is convex. Toussaint [7] calls this property *weakly monotonic in the convex sense*. Obviously, for convex polyhedra, any vector $\vec{d}$ gives only convex cross-sections during the sweep. For many non-convex polyhedra no such vector exists. We give an $O(n \log n)$ time algorithm to find a vector $\vec{d}$ if one exists, for a simple polyhedron with n vertices. In case we allow more than one convex polygon in the cross-section, but no reflex vertices, we solve the problem in linear time.

The second question deals with cross-sections of simple polyhedra that are always simply-connected. This property is called *weakly monotonic* [7]. Again the problem is to determine a vector $\vec{d}$, if one exists, such that any plane normal to $\vec{d}$ intersects P in a simple polygon. This cross-section may degenerate into a line segment, single point, or be empty. The cross-section may not become disconnected, nor may it contain a hole. We solve the problem in $O(n^2)$ time.

Thirdly, we consider sweeping a simple polygon with a line, but we allow the line to change its orientation. The problem is to determine if such a sweep exists that passes over the polygon P, such that every cross-section is connected (generally, a single line segment). The problem is solved in quadratic time, also if we require additionally that the sweep line never goes back over any point of P.

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2 Sweeping a simple polyhedron with convex cross-sections

Given a simple polyhedron P with n vertices in 3-space, we want to determine if there is a sweep direction $\vec{d}$ so that every cross-section is a convex polygon. A slight variation to this question is deciding whether every cross-section contains zero or more convex polygons, but never reflex vertices. We call such a polyhedron sweepable in direction $\vec{d}$. We will first establish that we only have to consider reflex edges of the polyhedron in order to solve these problems.

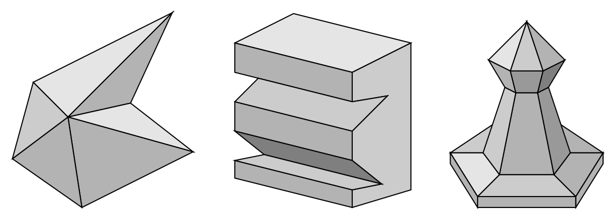

Figure 1: Polyhedra with convex cross-sections only.

Lemma 1 (proof omitted) *Let P be a simple polyhedron and let E_r be the set of its reflex edges. P is sweepable in direction $\vec{d}$ such that every cross-section is empty, a point, a line segment, or a (collection of) convex polygon(s) if and only if the sweep plane with normal $\vec{d}$ is parallel to all edges in E_r.*

Note that there can be cases where the cross-section consists of more than one convex polygon, but never a reflex vertex. This can occur when the sweep plane contains a reflex edge, and both facets incident to the reflex edge are to the same side of the sweep plane. Figure 1 shows some polyhedra that are sweepable, where certain sweep directions only give single convex polygons as the cross-section, whereas other sweep directions may give several convex polygons in some cross-section. Note that all essential changes to the cross-section occur when the sweep plane reaches or passes a vertex.

To decide algorithmically if a direction exists with only convex polygons as intersections, we distinguish four cases:

1. Polyhedron P is convex.
2. Polyhedron P has at least one reflex edge, and all reflex edges have the same orientation.
3. Polyhedron P has at least two reflex edges, and all reflex edges of P have orientations that span a plane.
4. Polyhedron P has three reflex edges that are linearly independent.

It is simple to determine in linear time which of the four cases applies for a given polyhedron. Cases 1 and 4 can be handled trivially: In case 1 any sweep direction works, and in case 4 no sweep direction works, which follows from Lemma 1.

Cases 2 and 3 are also trivial if collections of convex polygons are allowed in the cross-section, but no reflex vertices. The answer is positive, again by the lemma.

For the problem where the intersection must always be at most one convex polygon, an additional test is required. All directions such that the sweep plane contains a reflex edge and both incident faces are to the same side, are also forbidden then. If case 3 above applies, there is only one candidate orientation for a sweep plane, and we can test the extra condition easily in linear time. If case 2 applies, the sweep direction must be normal to the reflex edge(s) of P, which implies that the sweep plane still has one rotational degree of freedom. This can be represented by a circle of candidate sweep directions. Each reflex edge eliminates two antipodal intervals (circular arc) from the circle of orientations to fulfill the condition that both incident faces may not be on the same side of the sweep plane when the sweep plane contains that edge. To test if all circular arcs together cover the circle, implying that no good sweep direction exists, we solve the problem by sorting the endpoints of the circular arcs along the circle in $O(n \log n)$ time.

Theorem 2 *Let P be a simple polyhedron with n vertices. In linear time, one can test if a sweep direction exists so that every cross-section is a collection of convex polygons. In $O(n \log n)$ time, one can test if a sweep direction exists so that every cross-section is a single convex polygon.*

3 Sweeping a simple polyhedron with simply-connected cross-sections

In this section we wish to determine whether a simple polyhedron P admits a sweep direction so that every cross-section is simply-connected. To this end, we analyze what topological changes can occur when sweeping a simple polyhedron by a plane in a specified direction. Assume we sweep vertically, from top to bottom, with a horizontal plane. The possible changes when sweeping in direction $-\vec{z}$ are (see Figure 2):

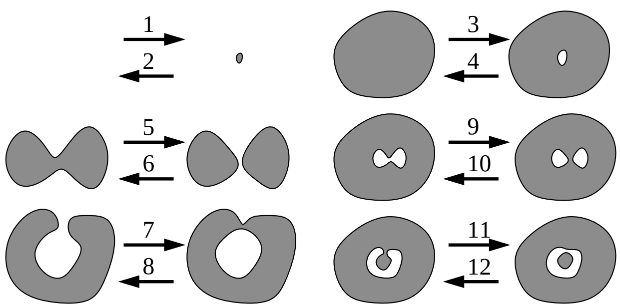

Figure 2: Topological changes that can occur in the cross-sections.

1. A new component in the cross-section starts. This is caused by a local maximum of P in direction $\vec{z}$.
2. A component in the cross-section disappears. This is caused by a local minimum of P in direction $\vec{z}$.
3. A component in the cross-section creates a hole 'in the middle'. This is caused by a local maximum of the complement of P.
4. A component in the cross-section closes a hole 'in the middle'. This is caused by a local minimum of the complement of P.
5. A component splits into two components. This is caused by a saddle-type vertex.
6. Two components touch to form one component. This is caused by a saddle-type vertex.
7. A component in the cross-section creates a hole 'at the boundary', that is, the outer boundary of a component curls to touch itself, thereby creating a hole. This is caused by a saddle-type vertex.
8. A component in the cross-section opens a hole 'at the boundary'.
9. A hole of a component splits into two.
10. Two holes touch to form one hole.
11. A hole has a new component split off inside the hole.
12. A component inside a hole merges with a surrounding component.

All of these changes can only occur when the sweep plane reaches or passes a vertex. We represent all possible sweep directions by a sphere centered at the origin o. A point q on the sphere represents the sweep direction $\vec{oq}$. Each vertex of polyhedron P defines a partition of the sphere of directions into regions where some topological change occurs in the cross-section. A vertex can be a local maximum in a certain sweep direction; this was studied and used for the application of mold filling in [1]. The same vertex can be one where—in another sweep direction—a hole is created or destroyed in the cross-section. In yet another direction it may cause no topological change at all. See Figure 3.

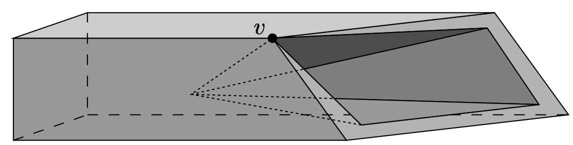

Figure 3: Depending on the sweep direction, different topological changes can occur in the cross-section when the sweep plane reaches v.

On the sphere of directions, the boundaries between different regions defined by a vertex are determined by the directions of the edges incident to the vertex. Every edge in fact gives rise to a great circle on the sphere of directions which represents the collection of normals of the sweep plane for which it is parallel to the edge. In other words, the circle represents the perpendiculars to the edge.

Since a polyhedron with n vertices has $O(n)$ edges, the sphere of directions is partitioned into $O(n^2)$ regions bounded by pieces of great circles. Each region defines a set of directions for which a sweep encounters the same topological changes. We can compute this arrangement by standard methods in quadratic time.

Note that a sweep in a direction $\vec{d}$ has at most one single simple polygon in the cross-section if and only if it has one local maximum and one local minimum in that direction, and no other topological changes occur. The number of local maxima and minima, and the occurrence of topological changes are interdependent. A polyhedron that has one local maximum and two local minima must have a vertex at which the cross-section is split, one part of the cross-section ending at the one local minimum and the other ending at the other.

Lemma 3 (proof omitted) *Given a simple polyhedron P and a sweep direction $\vec{d}$, every cross-section with a plane normal to $\vec{d}$*

is empty or simply-connected if and only if P has exactly one local maximum and one local minimum in direction $\vec{d}$, and no local maximum or minimum of the exterior in direction $\vec{d}$.

To determine a sweep direction in which every cross-section is simply-connected, we only have to consider local maxima and minima of the interior and the exterior of P. Every vertex determines a (possibly empty) region on the sphere of directions such that it is a local maximum (or minimum) for those sweep directions. In the same way as in [1], this can be done in quadratic time. We determine a cell in the arrangement on the sphere of directions that corresponds to the smallest number of local maxima plus minima (interior and exterior). If there is one local maximum and one local minimum of the interior, this smallest number is 2, and only then is the polyhedron sweepable with simply-connected cross-sections.

Theorem 4 *Let P be a simple polyhedron with n vertices. In quadratic time, one can test if a sweep direction exists so that every cross-section is simply connected or empty.*

4 Sweeping a simple polygon in varying directions

We go down in dimension to the planar case and consider sweeping with a line, but now the sweep line is allowed to change its direction during the sweep. We model the movements of the sweep line by a sequence of translations and rotations that alternate. For rotations we must specify a center of rotation on the sweep line, and an angle of rotation. We start out with a simple polygon P in the plane, and a directed horizontal line strictly above all vertices of the polygon. The final situation of the sweep is one where the directed line is strictly below all vertices of the polygon. The line must be directed, otherwise it could go from initial to final position using a half turn beside the polygon, not intersecting it at all. The simple polygon

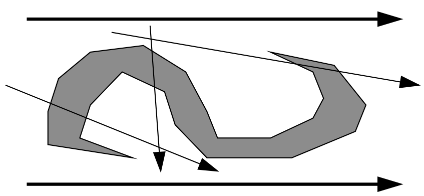

Figure 4: The top line shows the position before the sweep; the bottom line after the sweep. Three possible intermediate positions are shown.

P is assumed to be a closed set with interior and boundary. During the translations and rotations, the sweep line may only intersect P in a single line segment, or a point, or not at all. So the sweep line (generally) intersects the boundary of P in two points. Figure 4 shows that this kind of sweep is more general than sweeping by translation only. We will study two versions of the sweep problem with varying directions. One where the sweep line is not allowed to go back and one where this is allowed. For each case we develop an algorithm to decide if the polygon is sweepable in the desired way. Note that this is related to but different from *walkable polygons* [5] and *street polygons* [6].

The key to the solution is dualization. We require that the sweep line always intersect the polygon in at most two edges, except when going over vertices. In the dual, the edges of the polygon are double wedges, and the sweep line is a moving point. Translation of the sweep line implies that the dual 'sweep point' translates vertically, and rotation of the sweep line implies that the sweep point translates along a non-vertical line. When the sweep line goes from one edge to the next over a vertex, the sweep point goes from one double wedge to another via their common bounding line.

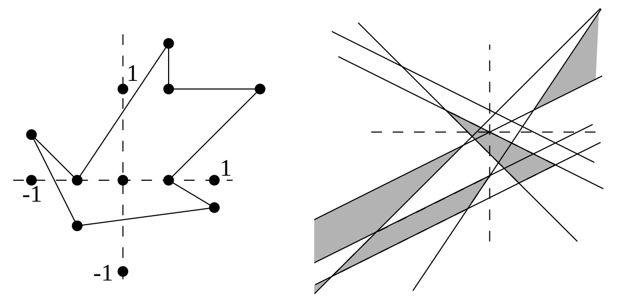

Figure 5: A simple polygon and the dual arrangement of its vertices. Forbidden faces are gray; two valid paths—through white faces only—are shown

The vertices of the polygon dualize to all relevant lines in the dual plane. The bottommost face is the one where the sweep point starts, and the topmost face is the one where it must end. If the dual sweep point lies inside more than two double wedges, then there are more than two edges of the polygon that intersect the sweep line. So the question is whether the sweep point can go from the bottommost face to the topmost face via faces that lie in at most two double wedges. From one face, the sweep point may only go to adjacent faces that share an edge in their boundary, not if they share a vertex.

Two lines at the same position but opposite orientation should not dualize to the same point. To deal with this, we use two copies of the dual plane, one for leftward directed lines and one for rightward directed lines, see Figure 6. The two arrangements are called $\mathcal{A}_1$ and $\mathcal{A}_2$; they are identical. The sweep is a path for a point in $\mathcal{A}_1$ and $\mathcal{A}_2$. Within one arrangement, the point dual to the sweep line can go from one face to an adjacent face over a shared edge. Any unbounded face in $\mathcal{A}_1$ is connected to an unbounded face in $\mathcal{A}_2$, namely, the one on the opposite side of the arrangement. The point dual to the sweep line can move between these faces, which corresponds to the situation where the sweep line rotates past the vertical direction.

At the start of the algorithm, we first rotate P slightly so that no two vertices have the same x-coordinate. This makes sure that we do not have parallel lines in the dual arrangements to be computed. Each whole dual arrangement can be computed with standard techniques in quadratic time, and we can also predetermine with every face in how many double wedges it lies (Chapter 8 in [2]). Faces in more than two double wedges are forbidden. Then the question whether P is sweepable becomes that of going from the bottom face of $\mathcal{A}_1$ to the top face of $\mathcal{A}_1$ via a sequence of adjacent faces that are not forbidden. Answering this question in the dual arrangements is easy; it simply is a depth-first search on the faces. The induced graph $\mathcal{G}$ has $O(n^2)$ nodes and arcs and is planar. This solves the version of sweeping where the sweep line is allowed to go back over vertices in $O(n^2)$ time in total.

Next we consider the version of the problem where the sweep line may not go back over any point of P. It is possible that the

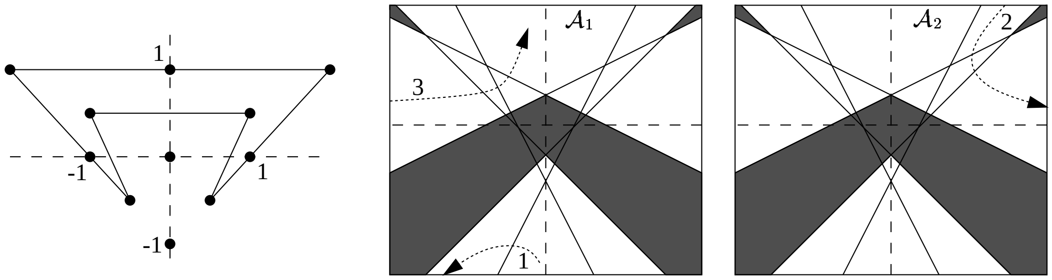

Figure 6: A simple polygon and the dual arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$. Any valid sweep over the polygon (for example, the one shown by the arrows 1, 2, 3) must go via faces of both arrangements.

sweep line needs to go back over part of an edge, but without going back over a vertex of P again. Hence, the problem cannot be solved on the adjacency graph $\mathcal{G}$ of the dual arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ that we used before. We use a different, refined arrangement where going backwards implies crossing a line of the arrangement again. We take all extensions of edges at reflex vertices inside P, and place an extra vertex where the extension hits the boundary of P.

Lemma 5 *Let P be a simple polygon, and let P' be P extended with the vertices obtained when edges are extended at all reflex vertices. If P' allows a sweep without going back over vertices, then P allows a sweep without going back at all.*

The algorithm is as follows. We start by computing the extensions of all edges at reflex vertices by preprocessing P for ray shooting queries [4]. This gives us the simple polygon P' with $O(n)$ extra vertices on the edges in $O(n \log n)$ time; let $|P'| = m$. Next, we rotate P' to make sure that no two vertices of P' have the same x-coordinate. We dualize the vertices of P', construct the arrangements $\mathcal{A}'_1$ and $\mathcal{A}'_2$ and the forbidden faces in these arrangements as before, and find a path from the lowest face to the highest face Then we determine if there is a path from the lowest face in $\mathcal{A}'_1$ to the highest face in $\mathcal{A}'_1$ through adjacent faces in $\mathcal{A}'_1$ and $\mathcal{A}'_2$ that crosses exactly m lines. If such a path exists, then clearly the corresponding motion of the sweep line doesn't go back over vertices. Otherwise, no motion of the desired type exists. Finding a shortest path in a graph with $O(m^2)$ nodes and arcs takes $O(m^2)$ time because the graph is planar [3].

Theorem 6 *Given a simple polygon P with n vertices, in $O(n^2)$ time we can determine whether a sweep with a line ℓ over P exists such that ℓ intersects P in at most one connected component. The same bound holds if, additionally, the sweep line is not allowed to go over any point of P more than once.*

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